On trajectory attractor approximations of the 3D Navier-Stokes system by various hydrodynamical alpha-models

V.V.Chepyzhov (IITP, RAS)

Infinite-dimensional dynamics, dissipative systems, and attractors

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Plan of the presentation

1. Introduction
2. The α-models
3. Trajectory attractor of the 3D NS system
4. Trajectory attractors of α-model
5. Trajectory convergence of α-models as $\alpha \to 0+$
6. On minimal limits of trajectory attractors as $\alpha \to 0+$
7. Conclusion

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§1. Introduction

An α-model is a mollification of the Navier–Stokes (NS) system in which the smoothing is performed by filtering of the velocity arguments in the bilinear term of the original NS system. The parameter α reflects the length scale of the model. Usually, the Green function associated with the Helmholtz operator $I - \alpha^2\Delta$ is considered as a filtering kernel with spatial width α.

The Lagrangian averaged Navier–Stokes-α (LANS-α) model (also known in the literature as the viscous 3D Camassa–Holm equations) was probably the first such α-model. It was demonstrated analytically and computationally in many works that the LANS-α model gives a good approximation in the study of problems related to turbulent flows.

Along the same lines, other approximate α-models were proposed and studied: Leray-α, Clark-α, simplified Bardina-α, modified Leray-α and other models.

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In many works, the Cauchy problems for all these models were studied, the global existence and uniqueness of weak (and strong) solutions were established, the smoothing properties of solutions were proved, the global attractors for the corresponding semigroups were constructed and the number of degree of freedom (the dimension of the global attractors) of these dynamical systems were estimated in terms of the relevant physical parameters.

It is needless to say that the analogous questions addressed to the 3D NS system remain without answers since the uniqueness theorem for the (existing) global weak solutions of the 3D NS system is not proved yet and so the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D NS system.

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However, the theory of trajectory attractors for evolution PDEs was developed with emphasis on the equations for which the uniqueness theorem of solutions of the corresponding initial-value problems does not hold or is not proved, e.g., for the 3D NS system.

In the present work, we study the connection between the long-time dynamics of solutions of various 3D α-models and the solutions of the exact 3D NS system as $\alpha \to 0+$ using the trajectory attractors approach.

We also suggest a simple classification of α-models. We partition the considered α-models into 2 classes: Class I and Class II depending on the orthogonal properties of the mollifying nonlinear terms. The trajectory attractors for the Class I α-model (e.g., the Leray-α model) converges "stronger" as $\alpha \to 0+$ to the limit than for the Class II α-models (e.g., the LANS-α model).

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§2. The α-models

We consider the following system:

$$\partial_t v = -\nu A v - PF(u,v) + g(x),\ \nabla \cdot v = 0, \quad (1)$$

$$v = u - \alpha^2 \Delta u, \quad \nabla \cdot u = 0, \quad (2)$$

$$x \in \mathbb{T}^3 := [\mathbb{R} \bmod 2\pi]^3,\ t \geq 0;$$

where $\nu > 0$ is the viscosity and P is the Leray projector.
In system (1), (2), the vector fields

$$v = (v^1(x,t), v^2(x,t), v^3(x,t)) \text{ and } u = (u^1(x,t), u^2(x,t), u^3(x,t))$$

are unknown and the external force $g = (g^1(x), g^2(x), g^3(x))$ is given. We assume that these functions have zero means:

$$\int_{\mathbb{T}^3} u(x,t)dx = 0,\ \int_{\mathbb{T}^3} v(x,t)dx = 0,\ \int_{\mathbb{T}^3} g(x)dx = 0.$$

We denote by $\mathscr{V}$ the space of trigonometrical polynomials $y(x) = (y^1(x), y^2(x), y^3(x))$ such that $\nabla \cdot y = 0$ and $\int_{\mathbb{T}^3} y(x)dx = 0$.

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We denote by $H^s, s \in \mathbb{R}_+$, the closure of the set $\mathscr{V}$ in the norms $\|\cdot\|_s$ of the space $H^s(\mathbb{T}^3)^3$. We set $H = H^0$ and the space $H^{-s}, s \geq 0$, is the dual to H^s. Then we consider the Leray orthogonal projector $P : L_2(\mathbb{T}^3)^3 \to H$. The Stokes operator $A = -P\Delta$ with domain $\mathscr{D}(A) = H^2$ is self-adjoint and positive.

The nonlinear term $F(u,v) = (F^1(u,v), F^2(u,v), F^3(u,v))$ in (1) has the vector components of the form

$$F^i(u,v) = \sum_{k,j,n=1}^{3} C^i_{kjn} u^k \partial_{x_j} v^n + D^i_{kjn} v^k \partial_{x_j} u^n + E^i_{kjn} u^k \partial_{x_j} u^n, \qquad (3)$$

where C^i_{kjn}, D^i_{kjn}, and E^i_{kjn} are some real coefficients. Note that in (3), we do not use monomials of the form $v^k \partial_{x_j} v^n$ since they do not contain the components of the "mollifying" vector field u.

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We shall consider various operators $F(u,v)$ of the form (3) which correspond to various α-models, that are characterized by the following two basic properties.

First poperty. We assume that for $u = v \in H^1$

$$PF(v,v) = P\sum_{j=1}^{3} v^j \partial_{x_j} v = P(v\cdot\nabla)v$$

is the nonlinear term from the classical 3D NS system.

That is, for $\alpha = 0$, the system (1), (2) reads

$$\partial_t v = -\nu A v - P(v\cdot\nabla)v + g(x),\ x \in \mathbb{T}^3,\ t \geq 0,$$

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Second property of orthogonality for the nonlinear term $F(u,v)$ partitions α-models into two classes.

CLASS I. We assume that

$$\langle F(u,v), v\rangle = 0,\ \forall u,v \in H^1. \tag{4}$$

CLASS II. We assume that

$$\langle F(u,v), u\rangle = 0,\ \forall u,v \in H^1. \tag{5}$$

We now consider examples of α-models of these classes.

Class I. The Leray-α model, $F(u,v) = (u\cdot\nabla)v$.
It is well known form the theory of the NS system that

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so identity (4) holds. For $u = v$, we clearly have the exact 3D NS system.

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$$\langle F(u, v), u\rangle = 0, \ \forall u, v \in H^1. \tag{5}$$

We now consider examples of α-models of these classes.

Class I. The Leray-α model, $F(u, v) = (u \cdot \nabla)v$.
It is well known form the theory of the NS system that

$$\langle (u \cdot \nabla)v, v\rangle = 0, \ \forall u, v \in H^1$$

so identity (4) holds. For $u = v$, we clearly have the exact 3D NS system.

Class II. The LANS-α model, $F(u,v) = -u \times (\nabla \times v)$.
It is easy to show that

$$\langle u \times (\nabla \times v), u \rangle = 0, \ \forall u, v \in H^1$$

so identity (5) holds. It follows from the vector calculus that the nonlinear term $-u \times (\nabla \times v)$ satisfies:

$$-u \times (\nabla \times v) = (u \cdot \nabla) v + \sum_{j=1}^{3} u^j \nabla v^j$$

and then, for $v = u$,

$$-u \times (\nabla \times u) = (u \cdot \nabla) u + \frac{1}{2} \nabla (u \cdot u).$$

Consequently,

$$-P(u \times (\nabla \times u)) = P(u \cdot \nabla) u,$$

since the operator P project any gradient vector to zero. Thus, for $u = v$, we also obtain the 3D NS system.

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Two more α-models appear if, in the previous examples, we interchange variables u and v. Under this procedure, the α-models change classes and we obtain the following examples.

Class II. The modified Leray-α: $F(u,v) = (v \cdot \nabla)u$.

Class I. The modified LANS-α: $F(u,v) = -v \times (\nabla \times u)$.

The same trick clearly changes class for any α-model. That is, if $F_I(u,v)$ belongs to Class I, then $F_{II}(v,u) := F_I(u,v)$ belongs to Class II and vice versa.

In all the above examples, α-models were constructed by the mollification of the only one argument in the nonlinear term of the 3D NS. Mollifying both of them, one gets so-called the simplified Bardina-α model having the class II.

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In conclusion, we note that there are infinitely many α-models of both classes. Indeed, if we have two different α-models from the same class with functions $F_1(u,v)$ and $F_2(u,v)$, then the function

$$F(u,v) = \theta F_1(u,v) + (1-\theta)F_2(u,v),\ \forall \theta \in (0,1),$$

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§3. Trajectory attractor of the 3D NS system

Let $g \in H$. We consider the family $\mathscr{K}^+$ of all weak solutions $\{v(t), t \geq 0\}$ of the 3D NS system

$$\partial_t v = -\nu A v - P(v \cdot \nabla)v + g(x),\ x \in \mathbb{T}^3,\ t \geq 0, \tag{6}$$

having the following properties:

1) $v(\cdot) \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H^1) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H)$,
2) $v(t)$ satisfies the following energy inequality:

$$-\frac{1}{2}\int_0^\infty \|v(s)\|_0^2 \psi'(s)ds + \nu \int_0^\infty \|\nabla v(s)\|_0^2 \psi(s)ds \leq \int_0^\infty \langle g, v(s)\rangle \psi(s)ds \tag{7}$$

for every positive scalar function $\psi(\cdot) \in C_0^\infty(\mathbb{R}_+; \mathbb{R}_+)$.

We note that any solution $v(t)$, $t \geq 0$, of the Cauchy problem for (6) with initial data $v(0) \in H$ resulting from the Galerkin approximation method belongs to $\mathscr{K}^+$. Consequently, the set $\mathscr{K}^+$ is non-empty and relatively large. The set $\mathscr{K}^+$ is called the trajectory space of the 3D NS system.

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Equation (6) implies that $\partial_t v \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H^{-2})$ for any $v \in \mathscr{K}^+$.

We now define the Banach space

$$\mathscr{F}_+^{\mathrm{b}} = \left\{ v(\cdot) \mid v(\cdot) \in L_2^{\mathrm{b}}(\mathbb{R}_+; H^1) \cap L_\infty(\mathbb{R}_+; H),\ \partial_t v(\cdot) \in L_2^{\mathrm{b}}(\mathbb{R}_+; H^{-2}) \right\}$$

with norm

$$\|v\|_{\mathscr{F}_+^{\mathrm{b}}} = \|v\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + \|v\|_{L_\infty(\mathbb{R}_+;H)} + \|\partial_t v\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-2})},$$

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Proposition 1

1) $\mathscr{K}^+ \subset \mathscr{F}_+^{\mathrm{b}}$;
2) for any function $v(\cdot) \in \mathscr{K}^+$, the following inequality holds

$$\|T(h)v(\cdot)\|_{\mathscr{F}_+^{\mathrm{b}}} \leq C_0 \|v(\cdot)\|^2_{L_\infty(0,1;H)} \exp(-\nu\lambda_1 h) + R_0, \ \forall h \geq 0, \quad (8)$$

where λ_1 is the first positive eigenvalue of the operator A and the constants C_0, R_0 depend only on ν, λ_1, and $\|g\|_0$.

The proof of this proposition is given, e.g., in the book:

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§3. Trajectory attractor of the 3D NS system

In this space, we define the local weak topology Θ_+^{loc} generated by the following weak convergence: by definition, a sequence $\{v_n\} \subset \mathscr{F}_+^{\text{loc}}$ converges to $v \in \mathscr{F}_+^{\text{loc}}$ in Θ_+^{loc} as $n \to +\infty$, if $v_n \rightharpoonup v$ in $L_{2,w}(0,M;H^1)$, $v_n \rightharpoonup v$ in $L_{\infty,*w}(0,M;H)$, and $\partial_t v_n \rightharpoonup \partial_t v$ in $L_{2,w}(0,M;H^{-2})$ as $n \to +\infty$ for any $M > 0$.

Notice that $\mathscr{F}_+^{\text{b}} \subset \Theta_+^{\text{loc}}$ and any ball

$$B_R = \left\{ v(\cdot) \in \mathscr{F}_+^{\text{b}} \mid \|v\|_{\mathscr{F}_+^{\text{b}}} \le R \right\} \text{ in } \mathscr{F}_+^{\text{b}}$$

is compact in the space Θ_+^{loc}. Moreover, the corresponding topological space is metrizable.

The translation semigroup $\{T(h)\} := \{T(h), h \ge 0\}$ is continuous in the topology Θ_+^{loc}. Besides, the trajectory space $\mathscr{K}^+$ is closed in Θ_+^{loc}. The semigroup $\{T(h)\}$ maps $\mathscr{K}^+$ to itself: $T(h)\mathscr{K}^+ \subset \mathscr{K}^+$ for all $h \ge 0$.

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§3. Trajectory attractor of the 3D NS system

Recall, a set $P \subset \mathscr{F}_+^{\mathrm{loc}}$ is called attracting, if, for any bounded (in $\mathscr{F}_+^{\mathrm{b}}$) set $B \subset \mathscr{K}^+$, the set

$$T(h)B \to P \text{ as } h \to +\infty \quad \text{in the topology } \Theta_+^{\mathrm{loc}}. \tag{9}$$

Notice that the following embedding is continuous:

$$\Theta_+^{\mathrm{loc}} \subset L_2^{\mathrm{loc}}(\mathbb{R}_+; H^{1-\delta}),\ \forall \delta,\ 0 < \delta \leq 1.$$

Therefore, we have strong convergence (9) in the space $L_2(0, M; H^{1-\delta})$ for any $M > 0$.

Definition 1

A set $\mathfrak{A} \subset \mathscr{K}^+$ is called the trajectory attractor of the semigroup $\{T(h)\}$ in the topology Θ_+^{loc}, if:
1) $\mathfrak{A}$ is bounded in $\mathscr{F}_+^{\mathrm{b}}$ and compact in Θ_+^{loc};
2) $\mathfrak{A}$ is strictly invariant: $T(h)\mathfrak{A} = \mathfrak{A}$ for all $h \geq 0$; 3) $\mathfrak{A}$ is an attracting set for the semigroup $\{T(h)\}$ on $\mathscr{K}^+$ in Θ_+^{loc}.

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Inequality (8) implies that the ball

$$B_{2R_0} = \left\{ v(\cdot) \in \mathscr{F}_+^{\mathrm{b}} \mid \|v\|_{\mathscr{F}_+^{\mathrm{b}}} \leq 2R_0 \right\}$$

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It was indicated above that the ball B_{2R_0} is a compact metric space. Hence, according to the known classical theorem on attractors of semigroups, the continuous semigroup $\{T(h)\}$ on $\mathscr{K}^+$ has a compact (in Θ_+^{loc}) trajectory attractor $\mathfrak{A} \subset \mathscr{K}^+ \cap B_{2R_0}$:

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CLASS I. The Cauchy problem of the system (1), (2) with $u(0) \in H^2$ has a unique solution in the space of functions

$$u \in L_2(0, M; H^3) \cap L_\infty(0, M; H^2) \text{ and } \partial_t u \in L_2(0, M; H^1), \ \forall M > 0.$$

For the corresponding function $v = (I + \alpha^2 A)u$, we have

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Using energy equality (10) and (11), we prove the following a priory estimates for solutions of α-models from Class I or II.

Proposition 2

If $u(t)$ is a solution of α-model (1), (2), then,
for Class I, the function $v(t) = (I + \alpha^2 A)u(t) \in \mathscr{F}_+^{\mathrm{b}}$ and

$$\|T(h)v(\cdot)\|_{\mathscr{F}_+^{\mathrm{b}}} \leq C_1 \|v(\cdot)\|^2_{L_\infty(0,1;H)} \exp(-\nu\lambda_1 t) + R_1; \qquad (12)$$

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§4. Trajectory attractors of α-model

Similar to the space $\mathscr{K}^+$, we define the trajectory space $\mathscr{K}_\alpha^+$ for a given α-model of Class I or Class II.

For Class I, $\mathscr{K}_\alpha^+$ consists of all functions

$$\mathscr{K}_\alpha^+ = \{v_\alpha(t) = (I + \alpha^2 A)u_\alpha(t) \mid u_\alpha(0) \in H^2\},$$

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§4. Trajectory attractors of α-model

Recall that any element of the trajectory space $\mathscr{K}_\alpha^+$ of Class I and II satisfies the energy equality (10) and (11), respectively.

The translation semigroup $\{T(h)\}$ acts on $\mathscr{K}_\alpha^+$. It is easy to prove that the space $\mathscr{K}_\alpha^+$ is closed in $\Theta_+^{\rm loc}$. Inequalities (12) and (13) imply that $\mathscr{K}_\alpha^+ \subset \mathscr{F}_+^{\rm b}$ and there exists an absorbing set of the semigroup $\{T(h)\}$ in $\mathscr{K}_\alpha^+$, bounded in $\mathscr{F}_+^{\rm b}$ and compact in $\Theta_+^{\rm loc}$.

Then, similar to Section 3, we establish the existence of the trajectory attractor $\mathfrak{A}_\alpha$ of the α-model [belonging to Class I or II] for $\alpha > 0$, that is, the set $\mathfrak{A}_\alpha \subset \mathscr{K}_\alpha^+$, $\mathfrak{A}_\alpha$ is bounded in $\mathscr{F}_+^{\rm b}$, compact in $\Theta_+^{\rm loc}$,

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§4. Trajectory attractors of α-model

Recall that any element of the trajectory space $\mathscr{K}_\alpha^+$ of Class I and II satisfies the energy equality (10) and (11), respectively.

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It follows from a priori estimates (12) and (13) that the trajectory attractors $\mathfrak{A}_\alpha$ belongs to the ball B in $\mathscr{F}_+^{\mathrm{b}}$ with radius R_1, therefore, they are uniformly (w.r.t. $\alpha \in (0,1]$) bounded in the space $\mathscr{F}_+^{\mathrm{b}}$.

In the next section, we formulate a technical lemma that is very important in the study of the limit behaviour of solutions of α-models as $\alpha \to 0+$.

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In the next section, we formulate a technical lemma that is very important in the study of the limit behaviour of solutions of α-models as $\alpha \to 0+$.

Lemma

Let sequences $\{u_n\} \subset \mathscr{F}_+^{\mathrm{b}}$ and $\{\alpha_n\} \subset \mathbb{R}_+$ be given such that $\alpha_n \to 0+$ $(n \to \infty)$. Denote

$$v_n(t) = (I + \alpha_n^2 A) u_n(t) \quad \text{and} \quad w_n(t) = (I + \alpha_n^2 A)^{1/2} u_n(t).$$

Assume EITHER $\{v_n(t)\}$ is bounded in $\mathscr{F}_+^{\mathrm{b}}$ and

$$v_n(t) \to u(t) \quad \text{as } n \to \infty \quad \text{in } \Theta_+^{\mathrm{loc}}$$

OR $\{w_n(t)\}$ is bounded in $\mathscr{F}_+^{\mathrm{b}}$ and

$$w_n(t) \to u(t) \quad \text{as } n \to \infty \quad \text{in } \Theta_+^{\mathrm{loc}}.$$

THEN $\{u_n(t)\}$ is also bounded in $\mathscr{F}_+^{\mathrm{b}}$ and

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Using this lemma, we prove the following main result.

Consider an arbitrary α-model from class I or class II with corresponding trajectory space $\mathscr{K}^+_{\alpha_n}$.

Theorem 1

Let sequences $\{z_{\alpha_n}(t)\} \subset \mathscr{K}^+_{\alpha_n}$ and $\{\alpha_n\} \subset \mathbb{R}_+$ be given such that $\{z_{\alpha_n}(t)\}$ is bounded in the space $\mathscr{F}^{\mathrm{b}}_+$, $\alpha_n \to 0+$ $(n \to \infty)$, and

$$z_{\alpha_n}(\cdot) \to z(\cdot) \quad \text{as } \alpha_n \to 0+ \quad \text{in } \Theta^{\mathrm{loc}}_+.$$

Then $z(t), t \geq 0$, is a weak solution of the 3D Navier–Stokes system that satisfies the energy inequality, that is, $z \in \mathscr{K}^+$.

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Theorem 2

Let, for Class I,

$$B_\alpha = \left\{ v_\alpha(t) = (I + \alpha^2 A) u_\alpha(t), t \geq 0 \right\}, \quad 0 < \alpha \leq 1,$$

and for Class II

$$B_\alpha = \left\{ w_\alpha(t) = (I + \alpha^2 A)^{1/2} u_\alpha(t), t \geq 0 \right\}, \quad 0 < \alpha \leq 1,$$

be a uniformly bounded in $\mathscr{F}_+^{\mathrm{b}}$ family of trajectories from $\mathscr{K}_\alpha^+$:

$$\|B_\alpha\|_{\mathscr{F}_+^{\mathrm{b}}} \leq R, \quad \forall \alpha \in]0,1].$$

Then

$$T(h) B_\alpha \to \mathfrak{A} \; (h \to +\infty, \; \alpha \to 0+) \; \text{ in } \Theta_+^{\mathrm{loc}}, \tag{14}$$

where $\mathfrak{A}$ is the trajectory attractor of the 3D NS system (6).

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Corollary

Trajectory attractors $\mathfrak{A}_\alpha$ of the α-model converge to the trajectory attractor $\mathfrak{A}$ of the 3D NS system as $\alpha \to 0+$:

$$\mathfrak{A}_\alpha \to \mathfrak{A} \quad (\alpha \to 0+) \quad \text{in } \Theta_+^{\mathrm{loc}}. \tag{15}$$

Since

$$\Theta_+^{\mathrm{loc}} \subset L_2^{\mathrm{loc}}(\mathbb{R}_+; H^{1-\delta}), \quad 0 < \delta \leq 1,$$

the convergence holds also in the strong metric of the space $L_2^{\mathrm{loc}}(0, M; H^{1-\delta})$ for every $M > 0$:

$$\mathrm{dist}_{L_2^{\mathrm{loc}}(0,M;H^{1-\delta})}(\mathfrak{A}_\alpha, \mathfrak{A}) \to 0 \ (\alpha \to 0+).$$

Here, $\mathrm{dist}_{\mathscr{M}}(X, Y)$ denotes the non-symmetric Hausdorff deviation of a set X from a set Y in the metric space $\mathscr{M}$.

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§5. Trajectory convergence of α-models

Notice that, for the Class II α-models (e.g., the LANS-α model), we can not state the convergence in terms of functions $v_\alpha = (I + \alpha^2 A)u_\alpha(t)$ since we have proved only weak a priory estimates. Thus, we conclude that the Class I α-models (e.g., the Leray-α model) provides "stronger" approximation of the 3D NS.

Due to Lemma, the assertions of Theorem 2 are also valid for uniformly bounded families $\tilde{B}_\alpha$ of smooth trajectories u_α of an α-model. Here, for Class I,

$$\tilde{B}_\alpha = (I + \alpha^2 A)^{-1} B_\alpha = \{u_\alpha(t),\, t \geq 0\},$$

for Class II,

$$\tilde{B}_\alpha = (I + \alpha^2 A)^{-1/2} B_\alpha = \{u_\alpha(t),\, t \geq 0\},$$

and all convergences hold in term of the mollified solutions $u_\alpha(t)$ in places of $v_\alpha(t)$ (Class I) or $w_\alpha(t)$ (Class II):

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§6. On minimal limits of trajectory attractors as $\alpha \to 0+$

Let $\mathfrak{A}_\alpha$ be the trajectory attractor of an α-model, $0 < \alpha \leq 1$. We have established that $\mathfrak{A}_\alpha \subset B$, where B is the ball in $\mathscr{F}_+^{\mathrm{b}}$ with radius R_1 that is independent of α.

It is clear that the trajectory attractor $\mathfrak{A}$ of the exact NS system also belongs to the ball B.

Recall that the ball B is compact in the topology Θ_+^{loc}. It follows from the Uryson compactness theorem that the subspace $B \cap \Theta_+^{\mathrm{loc}}$ equipped with topology Θ_+^{loc} is metrizable. We denote the corresponding metric in $B \cap \Theta_+^{\mathrm{loc}}$ by $\rho(\cdot,\cdot)$. The metric space itself, we denote by B_ρ. It is compact and complete.

Using these new notations, the result of the previous section can be written in the form

$$\mathrm{dist}_{B_\rho}(\mathfrak{A}_\alpha, \mathfrak{A}) \to 0+ \quad \text{as} \quad \alpha \to 0+ .$$

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Recall that the ball B is compact in the topology Θ_+^{loc}. It follows from the Uryson compactness theorem that the subspace $B \cap \Theta_+^{\mathrm{loc}}$ equipped with topology Θ_+^{loc} is metrizable. We denote the corresponding metric in $B \cap \Theta_+^{\mathrm{loc}}$ by $\rho(\cdot,\cdot)$. The metric space itself, we denote by B_ρ. It is compact and complete.

Using these new notations, the result of the previous section can be written in the form

$$\mathrm{dist}_{B_\rho}(\mathfrak{A}_\alpha, \mathfrak{A}) \to 0+ \quad \text{as} \quad \alpha \to 0+ .$$

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Using these new notations, the result of the previous section can be written in the form

$$\operatorname{dist}_{B_\rho}(\mathfrak{A}_\alpha, \mathfrak{A}) \to 0+ \quad \text{as} \quad \alpha \to 0+ .$$

Recall that the set $\mathfrak{A} \subset B_\rho$ is closed in B_ρ.

Definition 2

Let $\mathfrak{A}_{\min}$ be the minimal closed subset of $\mathfrak{A}$ that satisfies the above attracting property , i.e.,

$$\lim_{\alpha \to 0+} \mathrm{dist}_{B_\rho}(\mathfrak{A}_\alpha, \mathfrak{A}_{\min}) = 0$$

and $\mathfrak{A}_{\min}$ belongs to every closed subset $\mathfrak{A}' \subseteq \mathfrak{A}$ for which

$$\lim_{\alpha \to 0+} \mathrm{dist}_{B_\rho}(\mathfrak{A}_\alpha, \mathfrak{A}') = 0.$$

We call the set $\mathfrak{A}_{\min}$ the minimal limit of the trajectory attractors $\mathfrak{A}_\alpha$ *as* $\alpha \to 0+$.

§6. On minimal limits of trajectory attractors as $\alpha \to 0+$

Recall that the set $\mathfrak{A} \subset B_\rho$ is closed in B_ρ.

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Let $\mathfrak{A}_{\text{min}}$ be the minimal closed subset of $\mathfrak{A}$ that satisfies the above attracting property , i.e.,

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§6. On minimal limits of trajectory attractors as $\alpha \to 0+$

It is clear that the minimal limit $\mathfrak{A}_{\min}$ is unique. To prove the existence, it is easy to verify that the set

$$\mathfrak{A}_{\min} = \bigcap_{0<\delta\leq 1} \left[\bigcup_{0<\alpha\leq\delta} \mathfrak{A}_{\alpha} \right]_{B_\rho}.$$

satisfies all the needed properties.

We now formulate the final theorem of the report.

Theorem 3

For every α-model, the minimal limit $\mathfrak{A}_{\min}$ of the trajectory attractors $\mathfrak{A}_\alpha$ *as* $\alpha \to 0+$ is a connected component of the trajectory attractor $\mathfrak{A}$. Moreover, the set $\mathfrak{A}_{\min}$ is compact and strictly invariant with respect to the translation semigroup, that is,

$$T(h)\mathfrak{A}_{\min} = \mathfrak{A}_{\min}, \quad \forall h \geq 0.$$

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The concept of the minimal limit $\mathfrak{A}_{\min}$ of the trajectory attractors as $\alpha \to 0+$ was suggested by Mark Iosifovich Vishik. The properties of the minimal limit make it a very useful object in the study of various α-models that approximate the exact 3D NS system.

We note that the question of the connectedness of the trajectory attractor $\mathfrak{A}$ of the 3D NS system remains open.

Some years ago, Mark Iosifovich also has formulated the following hypothesis: to different α-models of the 3D NS system (Camassa–Holm, Leray-α, Clark-α, simplified Bardina-α, etc.), different minimal limits of their trajectory attractors $\mathfrak{A}_\alpha$ as $\alpha \to 0+$ may correspond.

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§7. Conclusion

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THANK YOU VERY MUCH !