Stability analysis of abstract systems of Timoshenko type

Filippo Dell'Oro

Institute of Mathematics of the Academy of Sciences of the Czech Republic

Infinite-Dimensional Dynamics, Dissipative Systems, and Attractors, Nizhny Novgorod - July 13-17, 2015

Joint work with

Valeria Danese (Politecnico di Milano)
Vittorino Pata (Politecnico di Milano)

Stability analysis of abstract systems of Timoshenko type

arXiv: 1504.00804

The abstract system

- $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$ infinite-dimensional separable real Hilbert space

The abstract system

- $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ infinite-dimensional separable real Hilbert space
- $A : \mathfrak{D}(A) \subset H \to H$ strictly positive selfadjoint linear operator

The abstract system

- $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ infinite-dimensional separable real Hilbert space
- $A : \mathfrak{D}(A) \subset H \to H$ strictly positive selfadjoint linear operator

$\longrightarrow$ the dense embedding $\mathfrak{D}(A) \subset H$ **need not be compact**

The abstract system

- $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ infinite-dimensional separable real Hilbert space
- $A : \mathfrak{D}(A) \subset H \to H$ strictly positive selfadjoint linear operator

$\longrightarrow$ the dense embedding $\mathfrak{D}(A) \subset H$ **need not be compact**

$$\begin{cases} \rho_1 \ddot{\varphi} + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi + \psi) = 0 \\ \rho_2 \ddot{\psi} + bA\psi + a(A^{\frac{1}{2}}\varphi + \psi) - \delta A^{\gamma}\theta = 0 \\ \rho_3 \dot{\theta} + cA\theta + \delta A^{\gamma}\dot{\psi} = 0 \end{cases} \tag{T}$$

The abstract system

- $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$ infinite-dimensional separable real Hilbert space
- $A : \mathfrak{D}(A) \subset H \to H$ strictly positive selfadjoint linear operator

  $\longrightarrow$ the dense embedding $\mathfrak{D}(A) \subset H$ **need not be compact**

$$\begin{cases} \rho_1 \ddot{\varphi} + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi + \psi) = 0 \\ \rho_2 \ddot{\psi} + bA\psi + a(A^{\frac{1}{2}}\varphi + \psi) - \delta A^{\gamma}\theta = 0 \\ \rho_3 \dot{\theta} + cA\theta + \delta A^{\gamma}\dot{\psi} = 0 \end{cases} \tag{T}$$

- Unknowns functions $\varphi, \psi, \theta : [0, \infty) \to H$

The abstract system

- $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$ infinite-dimensional separable real Hilbert space
- $A : \mathfrak{D}(A) \subset H \to H$ strictly positive selfadjoint linear operator

  $\longrightarrow$ the dense embedding $\mathfrak{D}(A) \subset H$ **need not be compact**

$$\begin{cases} \rho_1 \ddot{\varphi} + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi + \psi) = 0 \\ \rho_2 \ddot{\psi} + bA\psi + a(A^{\frac{1}{2}}\varphi + \psi) - \delta A^{\gamma}\theta = 0 \\ \rho_3 \dot{\theta} + cA\theta + \delta A^{\gamma}\dot{\psi} = 0 \end{cases} \tag{T}$$

- Unknowns functions $\varphi, \psi, \theta : [0, \infty) \to H$
- Coupling exponent $\gamma \in \mathbb{R}$

The abstract system

- $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ infinite-dimensional separable real Hilbert space
- $A : \mathfrak{D}(A) \subset H \to H$ strictly positive selfadjoint linear operator

  $\longrightarrow$ the dense embedding $\mathfrak{D}(A) \subset H$ **need not be compact**

$$\begin{cases} \rho_1 \ddot{\varphi} + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi + \psi) = 0 \\ \rho_2 \ddot{\psi} + bA\psi + a(A^{\frac{1}{2}}\varphi + \psi) - \delta A^{\gamma}\theta = 0 \\ \rho_3 \dot{\theta} + cA\theta + \delta A^{\gamma}\dot{\psi} = 0 \end{cases} \tag{T}$$

- Unknowns functions $\varphi, \psi, \theta : [0, \infty) \to H$
- Coupling exponent $\gamma \in \mathbb{R}$
- Structural parameters $\rho_1, \rho_2, \rho_3, a, b, c, \delta > 0$

For the particular choice $H = L^2(0, \ell)$ and

$$A = -\partial_{xx} \qquad \text{with domain} \qquad \mathfrak{D}(A) = H^2(0, \ell) \cap H_0^1(0, \ell)$$

(T) can be seen as a nonlocal **Timoshenko beam system**

For the particular choice $H = L^2(0,\ell)$ and

$$A = -\partial_{xx} \qquad \text{with domain} \qquad \mathfrak{D}(A) = H^2(0,\ell) \cap H_0^1(0,\ell)$$

(T) can be seen as a nonlocal **Timoshenko beam system**

$$\varphi, \psi, \theta : (x,t) \in [0,\ell] \times [0,\infty) \mapsto \mathbb{R}$$

represent the transverse displacement, the rotation angle of a filament and the relative temperature

For every fixed $\gamma \in \mathbb{R}$ system (T) is shown to generate a (linear) **contraction semigroup**

$$S(t) = \mathrm{e}^{\mathrm{tL}} : \mathcal{H} \to \mathcal{H}$$

acting on the natural weak energy space $\mathcal{H}$

For every fixed $\gamma \in \mathbb{R}$ system (T) is shown to generate a (linear) **contraction semigroup**

$$S(t) = \mathrm{e}^{\mathrm{tL}} : \mathcal{H} \to \mathcal{H}$$

acting on the natural weak energy space $\mathcal{H}$

- Analysis of the stability properties of $S(t)$ **in dependence of** γ

For every fixed $\gamma \in \mathbb{R}$ system (T) is shown to generate a (linear) **contraction semigroup**

$$S(t) = \mathrm{e}^{\mathrm{tL}} : \mathcal{H} \to \mathcal{H}$$

acting on the natural weak energy space $\mathcal{H}$

- Analysis of the stability properties of $S(t)$ **in dependence of** γ
- Crucial object is the **stability number**

$$\chi = \frac{a}{\rho_1} - \frac{b}{\rho_2}$$

  namely the difference between the propagation speeds of the first two hyperbolic equations

The main result

Theorem 1

The semigroup $S(t)$ is exponentially stable **if and only if**

$$\chi = 0 \qquad \textit{and} \qquad \gamma = \frac{1}{2}$$

The main result

Theorem 1

The semigroup $S(t)$ is exponentially stable **if and only if**

$$\chi = 0 \qquad \textit{and} \qquad \gamma = \frac{1}{2}$$

- We are **not** assuming the compactness of $\mathfrak{D}(A) \subset H$

The main result

Theorem 1

The semigroup $S(t)$ is exponentially stable **if and only if**

$$\chi = 0 \qquad \textit{and} \qquad \gamma = \frac{1}{2}$$

- We are **not** assuming the compactness of $\mathfrak{D}(A) \subset H$
  - $\rightarrow$ the **spectrum** $\sigma(A)$ can be a **complicated object** not simply made of an increasing sequence $\alpha_n \to \infty$ of eigenvalues

The main result

Theorem 1

The semigroup $S(t)$ is exponentially stable **if and only if**

$$\chi = 0 \qquad \textit{and} \qquad \gamma = \frac{1}{2}$$

- We are **not** assuming the compactness of $\mathfrak{D}(A) \subset H$
  - $\rightarrow$ the **spectrum** $\sigma(A)$ can be a **complicated object** not simply made of an increasing sequence $\alpha_n \to \infty$ of eigenvalues
- We establish a **general method** for the exponential stability analysis when $\sigma(A)$ consists of *approximate eigenvalues*

The main result

Theorem 1

The semigroup $S(t)$ is exponentially stable **if and only if**

$$\chi = 0 \qquad \textit{and} \qquad \gamma = \frac{1}{2}$$

- We are **not** assuming the compactness of $\mathfrak{D}(A) \subset H$
  - $\longrightarrow$ the **spectrum** $\sigma(A)$ can be a **complicated object** not simply made of an increasing sequence $\alpha_n \to \infty$ of eigenvalues
- We establish a **general method** for the exponential stability analysis when $\sigma(A)$ consists of *approximate eigenvalues*
  - $\longrightarrow$ revisit stability results for equations/systems already known when the leading operator has compact inverse

$$\begin{cases} \rho_1\ddot{\varphi} + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi + \psi) = 0 \\ \rho_2\ddot{\psi} + bA\psi + a(A^{\frac{1}{2}}\varphi + \psi) - \delta A^{\gamma}\theta = 0 \\ \rho_3\dot{\theta} + cA\theta + \delta A^{\gamma}\dot{\psi} = 0 \end{cases}$$

- The sole dissipation is the "thermal" one provided by the third equation

$$\begin{cases} \rho_1 \ddot{\varphi} + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi + \psi) = 0 \\ \rho_2 \ddot{\psi} + bA\psi + a(A^{\frac{1}{2}}\varphi + \psi) - \delta A^{\gamma}\theta = 0 \\ \rho_3 \dot{\theta} + cA\theta + \delta A^{\gamma}\dot{\psi} = 0 \end{cases}$$

- The sole dissipation is the "thermal" one provided by the third equation

  → **transfer** of thermal dissipation into mechanical one through the coupling

$$\begin{cases} \rho_1 \ddot{\varphi} + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi + \psi) = 0 \\ \rho_2 \ddot{\psi} + bA\psi + a(A^{\frac{1}{2}}\varphi + \psi) - \delta A^{\gamma}\theta = 0 \\ \rho_3 \dot{\theta} + cA\theta + \delta A^{\gamma}\dot{\psi} = 0 \end{cases}$$

- The sole dissipation is the "thermal" one provided by the third equation

  $\longrightarrow$ **transfer** of thermal dissipation into mechanical one through the coupling

- The coupling should be **sufficiently strong** in order for the third equation to transfer enough dissipation but **not too strong**

$$
\begin{cases}
\rho_1\ddot{\varphi} + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi + \psi) = 0 \\
\rho_2\ddot{\psi} + bA\psi + a(A^{\frac{1}{2}}\varphi + \psi) - \delta A^{\gamma}\theta = 0 \\
\rho_3\dot{\theta} + cA\theta + \delta A^{\gamma}\dot{\psi} = 0
\end{cases}
$$

- The sole dissipation is the "thermal" one provided by the third equation
  - $\rightarrow$ **transfer** of thermal dissipation into mechanical one through the coupling
- The coupling should be **sufficiently strong** in order for the third equation to transfer enough dissipation but **not too strong**
  - $\rightarrow$ wave equations with very strong damping are less likely to stabilize

$$
\begin{cases}
\rho_1 \ddot{\varphi} + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi + \psi) = 0 \\
\rho_2 \ddot{\psi} + bA\psi + a(A^{\frac{1}{2}}\varphi + \psi) - \delta A^{\gamma}\theta = 0 \\
\rho_3 \dot{\theta} + cA\theta + \delta A^{\gamma}\dot{\psi} = 0
\end{cases}
$$

- The sole dissipation is the "thermal" one provided by the third equation

  → **transfer** of thermal dissipation into mechanical one through the coupling

- The coupling should be **sufficiently strong** in order for the third equation to transfer enough dissipation but **not too strong**

  → wave equations with very strong damping are less likely to stabilize

- In addition the stability number χ comes into play

$$\begin{cases} \rho_1\ddot{\varphi} + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi + \psi) = 0 \\ \rho_2\ddot{\psi} + bA\psi + a(A^{\frac{1}{2}}\varphi + \psi) - \delta A^{\gamma}\theta = 0 \\ \rho_3\dot{\theta} + cA\theta + \delta A^{\gamma}\dot{\psi} = 0 \end{cases}$$

- The sole dissipation is the "thermal" one provided by the third equation

  → **transfer** of thermal dissipation into mechanical one through the coupling

- The coupling should be **sufficiently strong** in order for the third equation to transfer enough dissipation but **not too strong**

  → wave equations with very strong damping are less likely to stabilize

- In addition the stability number χ comes into play

  → when the waves exhibit **different speeds** the dissipation transfer **loses effectiveness**

Semiuniform stability

We are also interested in **weaker notions of stability**

We are also interested in **weaker notions of stability**

Definition 2

The semigroup $S(t)$ is said to be **semiuniformly stable** if there exists a nonnegative function $h(t)$ vanishing at infinity such that

$$\|S(t)z\|_{\mathcal{H}} \leq h(t)\|Lz\|_{\mathcal{H}} \quad \forall z \in \mathfrak{D}(L)$$

We are also interested in **weaker notions of stability**

Definition 2

The semigroup $S(t)$ is said to be **semiuniformly stable** if there exists a nonnegative function $h(t)$ vanishing at infinity such that

$$\|S(t)z\|_{\mathcal{H}} \leq h(t)\|Lz\|_{\mathcal{H}} \quad \forall z \in \mathfrak{D}(L)$$

Theorem 3

The following hold:

(i) $S(t)$ *is semiuniformly stable for* $\gamma \in \left[\frac{1}{2}, 1\right]$

(ii) $S(t)$ *is not semiuniformly stable when* $\gamma > 1$

Compact embedding

When the embedding $\mathfrak{D}(A) \Subset H$ **is compact** we are able to complete the picture

Compact embedding

When the embedding $\mathfrak{D}(A) \Subset H$ **is compact** we are able to complete the picture

Theorem 4

If $\mathfrak{D}(A) \Subset H$ *then* $S(t)$ *is semiuniformly stable if and only if* $\gamma \leq 1$

Compact embedding

When the embedding $\mathfrak{D}(A) \Subset H$ **is compact** we are able to complete the picture

Theorem 4

If $\mathfrak{D}(A) \Subset H$ then $S(t)$ is semiuniformly stable if and only if $\gamma \leq 1$

Theorem 5

If $\mathfrak{D}(A) \Subset H$ then $S(t)$ is stable (every trajectory goes to zero) for every $\gamma \in \mathbb{R}$

Sketch of the proof of Theorem 1

Sufficiency: **direct proof** based on energy-type functionals

Sketch of the proof of Theorem 1

Sufficiency: **direct proof** based on energy-type functionals

- Energy corresponding to the initial datum $z \in \mathcal{H}$

$$E(t) = \frac{1}{2}\|S(t)z\|_{\mathcal{H}}^2$$

Sketch of the proof of Theorem 1

Sufficiency: **direct proof** based on energy-type functionals

- Energy corresponding to the initial datum $z \in \mathcal{H}$

$$E(t) = \frac{1}{2}\|S(t)z\|_{\mathcal{H}}^2$$

- Auxiliary functionals

$$\Lambda_1(t) = 2\rho_2\langle\dot{\psi}(t), \psi(t)\rangle - 2\rho_1\langle\dot{\varphi}(t), \varphi(t)\rangle$$

$$\Lambda_2(t) = \frac{2\rho_2\rho_3}{\delta}\langle A^{-\frac{1}{2}}\theta(t), \dot{\psi}(t)\rangle$$

$$\Lambda_3(t) = 2\rho_2\langle\dot{\psi}(t), A^{\frac{1}{2}}\varphi(t) + \psi(t)\rangle - 2\rho_2\langle A^{\frac{1}{2}}\psi(t), \dot{\varphi}(t)\rangle$$

From $E(t)$ and $\Lambda_\imath(t)$ we construct a further functional $\Lambda(t)$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\Lambda + \nu E \leq 0$$

for some $\nu > 0$ and

$$\frac{1}{2}E(t) \leq \Lambda(t) \leq 2E(t)$$

From $E(t)$ and $\Lambda_\imath(t)$ we construct a further functional $\Lambda(t)$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\Lambda + \nu E \leq 0$$

for some $\nu > 0$ and

$$\frac{1}{2}E(t) \leq \Lambda(t) \leq 2E(t)$$

The condition $\chi = 0$ gives an **automatic cancellation** of terms otherwise impossible to handle

Necessity: we make use of the following well-know result

Necessity: we make use of the following well-know result

TFAE:

(i) $S(t)$ is exponentially stable

(ii) There exists $\varepsilon > 0$ such that

$$\inf_{\lambda \in \mathbb{R}} \|\mathrm{i}\lambda z - Lz\|_{\mathcal{H}} \geq \varepsilon \|z\|_{\mathcal{H}} \qquad \forall z \in \mathfrak{D}(L)$$

(iii) The imaginary axis $\mathrm{i}\mathbb{R}$ is contained in the resolvent set $\rho(L)$ of the operator L and

$$\sup_{\lambda \in \mathbb{R}} \|(\mathrm{i}\lambda - L)^{-1}\|_{L(\mathcal{H})} < \infty$$

Lemma 6

Let $\alpha \in \sigma(A)$ be fixed and let $\mathcal{Q} \subset \mathbb{R}$ be a given bounded set. Then, for every $\varepsilon > 0$ small enough, there exists a unit vector $w_\varepsilon \in H$ such that the vector

$$\xi_{q,\varepsilon} = A^q w_\varepsilon - \alpha^q w_\varepsilon$$

satisfies the relation

$$\|\xi_{q,\varepsilon}\| \leq \varepsilon \quad \forall q \in \mathcal{Q}$$

Select $\alpha_n \in \sigma(A)$ with $\alpha_n \to \infty$ (A is unbounded)

Select $\alpha_n \in \sigma(A)$ with $\alpha_n \to \infty$ (A is unbounded)

By Lemma 6, given a positive sequence $\nu_n \to 0$, there exist $w_n \in H$ such that the vectors

$$\xi_{q,n} = A^q w_n - \alpha_n^q w_n$$

fulfill the inequality

$$\|\xi_{q,n}\| \leq \nu_n \qquad \text{for } q = \gamma, \tfrac{1}{2}, 1$$

Select $\alpha_n \in \sigma(A)$ with $\alpha_n \to \infty$ (A is unbounded)

By Lemma 6, given a positive sequence $\nu_n \to 0$, there exist $w_n \in H$ such that the vectors

$$\xi_{q,n} = A^q w_n - \alpha_n^q w_n$$

fulfill the inequality

$$\|\xi_{q,n}\| \leq \nu_n \qquad \text{for } q = \gamma, \tfrac{1}{2}, 1$$

Next we set

$$\hat{z}_n = (0, c_1 w_n, 0, c_2 w_n, 0) \in \mathcal{H}$$

where the constants c_1, c_2 will be properly chosen in a later moment in such a way that

$$\|\hat{z}_n\|_{\mathcal{H}} = 1$$

Assume **by contradiction** that $S(t)$ is exponentially stable

Assume **by contradiction** that $S(t)$ is exponentially stable

Then, for any given sequence $\lambda_n \in \mathbb{R}$, the **resolvent equation**

$$\mathrm{i}\lambda_n z_n - L z_n = \hat{z}_n$$

has a unique solution

$$z_n = (\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, \theta_n) \in \mathfrak{D}(L)$$

Assume **by contradiction** that $S(t)$ is exponentially stable

Then, for any given sequence $\lambda_n \in \mathbb{R}$, the **resolvent equation**

$$\mathrm{i}\lambda_n z_n - L z_n = \hat{z}_n$$

has a unique solution

$$z_n = (\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, \theta_n) \in \mathfrak{D}(L)$$

Besides there is $\varepsilon > 0$ such that

$$\|z_n\|_{\mathcal{H}} \leq \frac{1}{\varepsilon}\|\hat{z}_n\|_{\mathcal{H}} = \frac{1}{\varepsilon}$$

namely **the sequence z_n is bounded**

Assume **by contradiction** that $S(t)$ is exponentially stable

Then, for any given sequence $\lambda_n \in \mathbb{R}$, the **resolvent equation**

$$\mathrm{i}\lambda_n z_n - L z_n = \hat{z}_n$$

has a unique solution

$$z_n = (\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, \theta_n) \in \mathfrak{D}(L)$$

Besides there is $\varepsilon > 0$ such that

$$\|z_n\|_{\mathcal{H}} \leq \frac{1}{\varepsilon}\|\hat{z}_n\|_{\mathcal{H}} = \frac{1}{\varepsilon}$$

namely **the sequence z_n is bounded**

→ We will reach a contradiction by showing **it is not so**

The solution $(\tilde{\varphi}_n, \tilde{\psi}_n, \theta_n)$ can be written in the form

$$\tilde{\varphi}_n = B_n w_n + q_n^1$$
$$\tilde{\psi}_n = C_n w_n + q_n^2$$
$$\theta_n = D_n w_n + q_n^3$$

The solution $(\tilde{\varphi}_n, \tilde{\psi}_n, \theta_n)$ can be written in the form

$$\begin{aligned} \tilde{\varphi}_n &= B_n w_n + q_n^1 \\ \tilde{\psi}_n &= C_n w_n + q_n^2 \\ \theta_n &= D_n w_n + q_n^3 \end{aligned}$$

for some $B_n, C_n, D_n \in \mathbb{C}$ and some vectors $q_n^\imath$ such that

$$q_n^\imath \perp w_n \qquad \text{for } \imath = 1, 2, 3$$

with

$$\|q_n^\imath\| \leq C$$

After some calculations, from the resolvent equation we obtain

$$-\rho_1\lambda_n^2 B_n + a[\alpha_n B_n + \sqrt{\alpha_n} C_n] = f_n + \mathrm{i}\lambda_n \rho_1 c_1$$

$$-\rho_2\lambda_n^2 C_n + b\alpha_n C_n + a[\sqrt{\alpha_n} B_n + C_n] - \mathrm{i}\lambda_n \delta \alpha_n^\gamma D_n = g_n + \mathrm{i}\lambda_n \rho_2 c_2$$

$$\mathrm{i}\lambda_n \rho_3 D_n + c\alpha_n D_n + \delta\alpha_n^\gamma C_n = h_n$$

After some calculations, from the resolvent equation we obtain

$$-\rho_1\lambda_n^2 B_n + a[\alpha_n B_n + \sqrt{\alpha_n} C_n] = f_n + \mathrm{i}\lambda_n \rho_1 c_1$$

$$-\rho_2\lambda_n^2 C_n + b\alpha_n C_n + a[\sqrt{\alpha_n} B_n + C_n] - \mathrm{i}\lambda_n \delta \alpha_n^{\gamma} D_n = g_n + \mathrm{i}\lambda_n \rho_2 c_2$$

$$\mathrm{i}\lambda_n \rho_3 D_n + c\alpha_n D_n + \delta \alpha_n^{\gamma} C_n = h_n$$

$$|f_n| \leq C\nu_n \qquad |g_n| \leq C(1 + |\lambda_n|)\nu_n \qquad |h_n| \leq C\nu_n$$

We distinguish **three cases**:

(i) $\gamma > \frac{1}{2}$

(ii) $\gamma \leq \frac{1}{2}$ and $\chi \neq 0$

(iii) $\gamma < \frac{1}{2}$ and $\chi = 0$

We distinguish **three cases**:

(i) $\gamma > \frac{1}{2}$

(ii) $\gamma \leq \frac{1}{2}$ and $\chi \neq 0$

(iii) $\gamma < \frac{1}{2}$ and $\chi = 0$

- In cases (i) and (ii) we show that

$$\|z_n\|_{\mathcal{H}} \geq \sqrt{\rho_1}\,\|\tilde{\varphi}_n\| \geq \sqrt{\rho_1}|B_n| \to \infty$$

We distinguish **three cases**:

(i) $\gamma > \frac{1}{2}$

(ii) $\gamma \leq \frac{1}{2}$ and $\chi \neq 0$

(iii) $\gamma < \frac{1}{2}$ and $\chi = 0$

- In cases (i) and (ii) we show that

$$\|z_n\|_{\mathcal{H}} \geq \sqrt{\rho_1}\,\|\tilde{\varphi}_n\| \geq \sqrt{\rho_1}|B_n| \to \infty$$

- In case (iii) we show that

$$\|z_n\|_{\mathcal{H}} \geq \sqrt{\rho_2}\,\|\tilde{\psi}_n\| \geq \sqrt{\rho_2}|C_n| \to \infty$$

Sketch of the proof of Theorem 3

- $S(t)$ is not semiuniformly stable when $\gamma > 1$ since its infinitesimal generator L is **not invertible**

Sketch of the proof of Theorem 3

- $S(t)$ is not semiuniformly stable when $\gamma > 1$ since its infinitesimal generator L is **not invertible**
- We want to prove that $S(t)$ is semiuniformly stable if $\gamma \in \left[\frac{1}{2}, 1\right]$

Sketch of the proof of Theorem 3

- $S(t)$ is not semiuniformly stable when $\gamma > 1$ since its infinitesimal generator L is **not invertible**
- We want to prove that $S(t)$ is semiuniformly stable if $\gamma \in \left[\frac{1}{2}, 1\right]$
- It is enough to show that no approximate eigenvalues of the operator L lie on the imaginary axis (Arendt, Batty et al.)

Assuming **by contradiction** that $\mathrm{i}\lambda \in \sigma_{\mathrm{ap}}(L)$ for some $\lambda \in \mathbb{R}$ there exists a sequence of unit vectors

$$z_n = (\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, \theta_n) \in \mathfrak{D}(L)$$

satisfying the relation

$$\mathrm{i}\lambda z_n - L z_n \to 0 \quad \text{in } \mathcal{H}$$

Assuming **by contradiction** that $\mathrm{i}\lambda \in \sigma_{\mathrm{ap}}(L)$ for some $\lambda \in \mathbb{R}$ there exists a sequence of unit vectors

$$z_n = (\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, \theta_n) \in \mathfrak{D}(L)$$

satisfying the relation

$$\mathrm{i}\lambda z_n - L z_n \to 0 \quad \text{in } \mathcal{H}$$

Componentwise

$$\mathrm{i}\lambda\varphi_n - \tilde{\varphi}_n \to 0 \quad \text{in } H^1$$

$$\mathrm{i}\lambda\rho_1\tilde{\varphi}_n + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi_n + \psi_n) \to 0 \quad \text{in } H$$

$$\mathrm{i}\lambda\psi_n - \tilde{\psi}_n \to 0 \quad \text{in } H^1$$

$$\mathrm{i}\lambda\rho_2\tilde{\psi}_n + A(b\psi_n - \delta A^{\gamma-1}\theta_n) + a(A^{\frac{1}{2}}\varphi_n + \psi_n) \to 0 \quad \text{in } H$$

$$\mathrm{i}\lambda\rho_3\theta_n + A(c\theta_n + \delta A^{\gamma-1}\tilde{\psi}_n) \to 0 \quad \text{in } H$$

Assuming **by contradiction** that $\mathrm{i}\lambda \in \sigma_{\mathrm{ap}}(L)$ for some $\lambda \in \mathbb{R}$ there exists a sequence of unit vectors

$$z_n = (\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, \theta_n) \in \mathfrak{D}(L)$$

satisfying the relation

$$\mathrm{i}\lambda z_n - L z_n \to 0 \quad \text{in } \mathcal{H}$$

Componentwise

$$\mathrm{i}\lambda\varphi_n - \tilde{\varphi}_n \to 0 \quad \text{in } H^1$$

$$\mathrm{i}\lambda\rho_1\tilde{\varphi}_n + aA^{\frac{1}{2}}(A^{\frac{1}{2}}\varphi_n + \psi_n) \to 0 \quad \text{in } H$$

$$\mathrm{i}\lambda\psi_n - \tilde{\psi}_n \to 0 \quad \text{in } H^1$$

$$\mathrm{i}\lambda\rho_2\tilde{\psi}_n + A(b\psi_n - \delta A^{\gamma-1}\theta_n) + a(A^{\frac{1}{2}}\varphi_n + \psi_n) \to 0 \quad \text{in } H$$

$$\mathrm{i}\lambda\rho_3\theta_n + A(c\theta_n + \delta A^{\gamma-1}\tilde{\psi}_n) \to 0 \quad \text{in } H$$

We reach a contradiction by showing that every single component of z_n goes to zero in its norm

Sketch of the proof of Theorem 4

- We want to prove that if A^{-1} is compact then $S(t)$ is semiuniformly stable when $\gamma < \frac{1}{2}$ as well

- We want to prove that if A^{-1} is compact then $S(t)$ is semiuniformly stable when $\gamma < \frac{1}{2}$ as well
- In this situation we can apply a famous result of Kato stating that

$$\sigma(L) = \sigma_{\mathrm{p}}(L)$$

Sketch of the proof of Theorem 4

- We want to prove that if A^{-1} is compact then $S(t)$ is semiuniformly stable when $\gamma < \frac{1}{2}$ as well
- In this situation we can apply a famous result of Kato stating that

$$\sigma(L) = \sigma_{\mathrm{p}}(L)$$

Therefore we only have to show that

$$\sigma_{\mathrm{p}}(L) \cap \mathrm{i}\mathbb{R} = \emptyset$$

Sketch of the proof of Theorem 4

- We want to prove that if A^{-1} is compact then $S(t)$ is semiuniformly stable when $\gamma < \frac{1}{2}$ as well
- In this situation we can apply a famous result of Kato stating that

$$\sigma(L) = \sigma_{\rm p}(L)$$

Therefore we only have to show that

$$\sigma_{\rm p}(L) \cap {\rm i}\mathbb{R} = \emptyset$$

The proof is again based on a suitable **contradiction argument**

THANK YOU