Parabolic equation of normal type connected with 3D Helmholtz system

A.V. Fursikov,
Moscow State University

International Conference-School
Infinite-dimensional Dynamics, Dissipative Systems and Attractors

Nizhny Novgorod, July 13, 2015

Navier-Stokes Equations (NSE)

$$\partial_t v(t,x) - \Delta v + (v,\nabla)v + \nabla p(t,x) = 0,$$

$$\text{div} v = 0,$$

$$v(t,\ldots,x_i,\ldots) = v(t,\ldots,x_i+2\pi,\ldots),\ i=1,2,3,$$

$$v(t,x)|_{t=0} = v_0(x)$$

Here $v(t,x) = (v_1,v_2,v_3)$ is a fluid velocity, $p(t,x)$ is a pressure.

Energy inequality:

$$\int_{\mathbb{T}^3} |v(t,x)|^2 dx + 2\int_0^t \int_{\mathbb{T}^3} |\nabla_x v(\tau,x)|^2 dx d\tau \leq$$

$$\int_{\mathbb{T}^3} |v_0(x)|^2 dx$$

Where $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ is 3D torus. Image of nonlinear operator $(v,\nabla)v$ at each point $v \in \Sigma \equiv \{u \in L_2 : \|u\|_{L_2} = 1\}$ is tangent to the sphere Σ, i.e. $v \perp_{L_2} (v,\nabla)v$

Helmholtz Equations

Curl of velocity

$$\omega(t,x) = \text{curl } v(t,x) =$$

$$= (\partial_{x_2} v_3 - \partial_{x_3} v_2,\ \partial_{x_3} v_1 - \partial_{x_1} v_3,\ \partial_{x_1} v_2 - \partial_{x_2} v_1)$$

Well-known formulas

$$(v,\nabla)v = \omega \times v + \nabla \frac{|v|^2}{2},$$

$$\text{curl } (\omega \times v) = (v,\nabla)\omega - (\omega,\nabla)v, \text{ if } \operatorname{div} v = \operatorname{div} \omega = 0$$

System of equations for curl

$$\partial_t \omega(t,x) - \Delta\omega + (v,\nabla)\omega - (\omega,\nabla)v = 0$$

$$\omega(t,x)|_{t=0} = \omega_0(x)$$

where $\omega_0 = \text{curl} v_0$

Function spaces for NSE and Helmholtz systems

Function spaces

$$V^m = V^m(\mathbb{T}^3) =$$

$$= \{v(x) \in (H^m(\mathbb{T}^3))^3 : \operatorname{div} v = 0,\ \int_{\mathbb{T}^3} v(x)dx = 0\}$$

where $H^m(\mathbb{T}^3)$ - is the Sobolev space. Using decomposition in Fourier series

$$v(x) = \sum_{k\in\mathbb{Z}^3} \widehat{v}(k)e^{ix\cdot k}, \quad \widehat{v}(k) = \int_{\mathbb{T}^3} \frac{v(x)}{(2\pi)^{-3}} e^{-ix\cdot k}dx,$$

where $x \cdot k = \sum_{j=1}^{3} x_j k_j,\ k = (k_1, k_2, k_3)$ and the formula curl curl $v = -\Delta v$, when $\operatorname{div} v = 0$, we get

$$\operatorname{curl}^{-1}\omega(x) = i \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{k \times \widehat{\omega}(k)}{|k|^2} e^{ix\cdot k}$$

Therefore operator

$$\operatorname{curl}\ :\ V^1 \longrightarrow V^0$$

realizes isomorphism of the spaces.

System of normal type and its derivation

Nonlinear term in Helmholtz equations

$$B(\omega) = (v, \nabla)\omega - (\omega, \nabla)v$$

The following formula holds

$$(B(\omega), \omega)_{V^0} = -\int_{\mathbb{T}^3} \sum_{j,k=1}^{3} \omega_j \partial_j v_k \omega_k dx \neq 0$$

and therefore

$$B(\omega) = B_n(\omega) + B_\tau(\omega),$$

where $B_n(\omega)$ is the component orthogonal to the sphere

$$\Sigma_\omega = \{u \in V^0 : \|u\|_{V^0} = \|\omega\|_{V^0}\}$$

at the point ω, and the vector $B_\tau(\omega)$ is tangent to Σ_ω at ω. It is clear that $B_n(\omega) = \Phi(\omega)\omega$ where Φ is unknown functional, that is determined from equation

$$\int_{\mathbb{T}^3} \Phi(\omega)\omega(x) \cdot \omega(x) dx = \int_{\mathbb{T}^3} (\omega(x), \nabla)v(x) \cdot \omega(x) dx$$

and has the form

$$\Phi(\omega) = \frac{\int_{\mathbb{T}^3}(\omega(x),\nabla)\text{curl }^{-1}\omega(x)\cdot\omega(x)dx}{\int_{\mathbb{T}^3}|\omega(x)|^2dx},\ \omega \neq 0,$$

$$\Phi(\omega) = 0, \qquad \omega \equiv 0$$

Normal parabolic system (NPS)

$$\partial_t\omega(t,x) - \Delta\omega - \Phi(\omega)\omega = 0, \quad \text{div}\,\omega = 0 \quad (1)$$

$$\omega(t,x)|_{t=0} = \omega_0(x) \quad (2)$$

Exact formula for NPS solution

Theorem 1. Let $S(t,x,y_0)$ - be solving operator for the Stokes system with periodic boundary conditions:

$$\partial_t y - \Delta y = 0,\ \text{div}\,y = 0,\ y|_{t=0} = y_0, \quad (3)$$

i.e. $S(t,x,y_0) = y(t,x)$. (We assume that $\text{div}\,y_0 = 0$). Then solution of the problem (1),(2) has the form

$$\omega(t,x;\omega_0) = \frac{S(t,x;\omega_0)}{1-\int_0^t \Phi(S(\tau,x;\omega_0))d\tau} \quad (4)$$

Unique solvability of NPS and continuity of solutions on initial conditions

Lemma 1. $\exists c > 0,\ \forall u \in V^{1/2} \quad \Phi(u) \le c\|u\|_{3/2}$

Lemma 2. $\forall \beta < 1/2 \quad \exists c_1 > 0 \quad \forall y_0 \in V^{-\beta}(\mathbb{T}^3)$,

$$\int_0^t \Phi(S(t,\cdot,y_0))dt \le c_1\|y_0\|_{-\beta}$$

Let $Q_T = (0,T) \times \mathbb{T}^3$, $T > 0$ or $T = \infty$. The space of solutions for NPS:

$$V^{1,2(-1)}(Q_T) = L_2(0,T;V^1) \cap H^1(0,T;V^{-1})$$

Moreover, we look for solutions $\omega(t,x;\omega_0)$ satisfying

Condition 1. If initial condition $\omega_0 \in V^0 \setminus \{0\}$ and solution $\omega(t,x;\omega_0) \in V^{1,2(-1)}(Q_T)$ then $\omega(t,\cdot,\omega_0) \ne 0\ \forall t \in [0,T]$

Theorem 2. For each $\omega_0 \in V^0$ there exists $T > 0$ such that there exists unique solution $\omega(t,x;\omega_0) \in V^{1,2(-1)}(Q_T)$ of the problem (1),(2) satisfying Condition 1.

Theorem 3. The solution $\omega(t,x;\omega_0) \in V^{1,2(-1)}(Q_T)$ of the problem (1),(2) depends continuously on initial condition $\omega_0 \in V^0$.

Properties of the functional Φ

On kernel of the functional $\Phi(S(t;u))$

Define the cone

$$K\Phi = \{u \in V^0 : \ \Phi(S(t;u)) \equiv 0 \quad \forall t \in \mathbb{R}_+\}$$

If $u \in K\Phi$ then $\lambda u \in K\Phi \ \ \forall \lambda \in \mathbb{R}$

Let

$$L = \{z \in V^0 : \ z(x) = \sum_{k \in \mathcal{U}} \hat{z}(k) e^{ik \cdot x}, \ \hat{z}(-k) = \overline{\hat{z}(k)}\},$$

where

$$\mathcal{U} = \{k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \backslash \{0\} : \ \sum_{j=1}^{3} k_j \text{ is odd}\}$$

Lemma 3. $L \subset K\Phi, \quad K\Phi \setminus L \neq \emptyset$

Structure of dynamical flow for NPS

$V^0(\mathbb{T}^3) \equiv V^0$ is phase space for problem (1),(2).

Definition 1. The set $M_- \subset V^0$ of ω_0, such that for solution $\omega(t,x;\omega_0)$ of problem (1),(2) satisfies inequality

$$\|\omega(t,\cdot;\omega_0)\|_0 \le \alpha \|\omega_0\|_0 e^{-t/2} \qquad \forall t > 0 \quad (*)$$

is called the set of stability. Here $\alpha > 1$ is a fixed number depending on $\|\omega_0\|_0$.

$$M_-(\alpha) = \{\omega_0 \in M_-;\ \omega(t,\cdot;\omega_0) \text{ satisfies } (*)\}$$

where $\alpha \ge 1$ is fixed. Then $M_- = \cup_{\alpha \ge 1} M_-(\alpha)$

If for $\omega_0 \in V^0$ the bound

$$\sup_{t \in \mathbb{R}_+} \int_0^t \Phi(S(\tau,\cdot;\omega_0))d\tau \le \frac{\alpha - 1}{\alpha}$$

holds then $\omega_0 \in M_-(\alpha)$.

Definition 2. The set $M_+ \subset V^0$ of ω_0, such that the corresponding solution $\omega(t, x; \omega_0)$ exists only on a finite time interval $t \in (0, t_0)$, and blows up at $t = t_0$ is called the set of explosions.

The formula holds:

$$M_+ = \{\omega_0 \in V^0 : \ \exists t_0 > 0 \ \int_0^{t_0} \Phi(S(\tau, \cdot; \omega_0))d\tau = 1\}$$

Definition 3. The set $M_g \subset V^0$ of ω_0, such that the corresponding solution $\omega(t, x; \omega_0)$ exists for time $t \in \mathbb{R}_+$, and $\|\omega(t, x; \omega_0)\|_0 \to \infty$ as $t \to \infty$ is called the set of growing.

Lemma 4. Sets M_-, M_+, M_g are not empty, and $M_- \cup M_+ \cup M_g = V^0$

Some subsets of unit sphere from V^0

Unit sphere: $\Sigma = \{v \in V^0 : \quad \|v\|_0 = 1\}$.
Subsets

$$A_-(t) = \{v \in \Sigma : \int_0^t \Phi(S(\tau,v))d\tau \le 0\},$$

$$A_0(t) = \{v \in \Sigma : \int_0^t \Phi(S(\tau,v))d\tau = 0\}$$

$$A_- = \cap_{t\ge 0} A_-(t),\ A_0 = \cap_{t\ge 0} A_0(t)$$

$$B_+ = \Sigma \setminus A_- \equiv$$

$$\equiv \{v \in \Sigma :\ \exists t_0 > 0\ \int_0^{t_0} \Phi(S(\tau,v))d\tau > 0\},$$

$$\partial B_+ = \{v \in \Sigma :\ \forall t > 0 \int_0^t \Phi(S(\tau,v))d\tau \le 0$$

$$\text{и } \exists t_0 > 0 :\ \int_0^{t_0} \Phi(S(\tau,v))d\tau = 0\}$$

On a structure of phase space

Important function on sphere Σ:

$$B_+ \ni v \to b(v) = \max_{t \geq 0} \int_0^t \Phi(S(\tau, v))d\tau \qquad (5)$$

Evidently, $b(v) > 0$ и $b(v) \to 0$ as $v \to \partial B_+$.
Let define the map $\Gamma(v)$:

$$B_+ \ni v \to \Gamma(v) = \frac{1}{b(v)} v \in V^0 \qquad (6)$$

It is clear that $\|\Gamma(v)\|_0 \to \infty$ as $v \to \partial B_+$.
The set $\Gamma(B_+)$ divides V^0 on two parts:

$$V_-^0 = \{v \in V^0 : \ [0, v] \cap \Gamma(B_+) = \emptyset\},$$

$$V_+^0 = \{v \in V^0 : \ [0, v) \cap \Gamma(B_+) \neq \emptyset\}$$

Let $B_+ = B_{+,f} \cup B_{+,\infty}$ where

$$B_{+,f} = \{v \in B_+ : \text{ max in (5) achives at } t < \infty\}$$

$$B_{+,\infty} = \{v \in B_+ : \text{ max in (5) does not achive at } t < \infty\}$$

Theorem 4. $M_- = V_-^0$, $M_+ = V_+^0 \cup B_{+,f}$, $M_g = B_{+,\infty}$

Feedback stabilization of equation with normal nonlinearity.

On the cylinder $\{(t,x) \in \mathbb{R}_+ \times \mathbb{T}^3\}$ where $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ is 3D torus we consider stabilization problem

$$\partial_t v(t,x) - \partial_{xx} v - \Phi(v)v = 0, \tag{7}$$

$$v|_{t=0} = v_0(x) + u_0(x) \tag{8}$$

where, recall,

$$\Phi(v) = \frac{\int_{\mathbb{T}^3} (v(x), \nabla)\text{curl}\ ^{-1} v(x) \cdot v(x) dx}{\int_{\mathbb{T}^3} |v(x)|^2 dx},\ v \neq 0,$$

$$\Phi(v) = 0, \qquad v \equiv 0, \tag{9}$$

$v_0(x)$ is a given initial vector field and $u(x)$ is a starting control supported on a cube

$$[-\rho,\rho]^3 \subset (-\pi,\pi]^3 := \mathbb{T}^3 \tag{10}$$

with arbitrary fixed $0 < \rho < \pi$ (in (10) we identify $(-\pi,\pi]^3$ with torus $\mathbb{T}^3$).

In other words for each $v_0 \in V^0$ one have to find a control $u_0 \in V^0$ supported in the cube (10) such that solution $v(t,x)$ of (7)(8) satisfies

$$\|v(t,\cdot)\|_0 \le \alpha e^{t/2}, \quad \text{as } t \to \infty$$

with some $\alpha > 0$.

We look for universal stabilizing control

$$u_0(x) = \lambda u(x), \quad \lambda \in \mathbf{R} \tag{11}$$

with

$$u(x) = \text{curl}(\xi_p(x_1,x_2,x_3)w(x_1,x_2,x_3),0,0) \tag{12}$$

where p is a natural number satisfying

$$\pi/(2p) \le \rho,$$

$\xi_p(x_1,x_2,x_3)$ is characteristic function of the cube $[-\pi/(2p), \pi/(2p)]^3 \subset \mathbb{T}^3$, and

$$w(x_1,x_2,x_3) = (2\sin x_1 \sin x_2 + \sin x_2 \sin x_3 + \sin x_1 \sin x_3)\Pi_{i=1}^3(1+\cos x_i) \tag{13}$$

Theorem. Given $v_0 \in M_+ \cup M_g$, $\rho > 0$ is small and fixed. There exists $u_0 \in L_2^0$ of the form (12), (13) such that $v_0 + u_0 \in M_-$.

The main step of proof consists of establishing inequality

$$\int_{\mathbb{T}^3} \Phi(S(t,x;u)dx \geq \beta e^{-18t} \quad \forall t \geq 0 \qquad (14)$$

with a positive constatnt β where $S(t,x,u)$ is the solution of heat equation with periodic boundary condition and initial condition $u(x)$ defined in (12),(13).

Using (14) it is possible to prove that

$$\forall v_0 \in M_+ \cup M_g \quad \exists \alpha > 1,\ \lambda_0 \gg 1 \quad \forall |\lambda| \geq \lambda_0$$

$$1 - \int_0^t \Phi(S(t,x,v_0 + \lambda u)dx \geq 1/\alpha \qquad (15)$$

In virtue of explicit formula (4) for solution of NPE (7) we get that

$$\|v(t,\cdot;v_0 + \lambda u)\|_{L_2}^2 \leq \alpha e^{-t}$$

This proves Theorem.

Remark Using result obtained in the Theorem one can prove nonlocal stabilization of Helmholtz system by feedback impulse control

$$\partial_t v - \partial_{xx} v - \Phi(v)v + B_\tau(v) = \sum_{j=1}^{N} \lambda_j u(x)\delta(t - t_j),$$

$$v|_{t=0} = v_0(x)$$

where $B_\tau(v)$ is tangential part of nonlinear operator for Helmholtz system. Here constants λ_j and time moments t_j are selected in dependence on some conditions connected with behavior of solution $v(t, \cdot)$.

Thank you

for attention