Attractors for the 2D damped Navier-Stokes system on large periodic domains and in $\mathbb{R}^2$

International Conference-School

Infinite-dimensional dynamics, dissipative systems, and attractors

Nizhny Novgorod University, July 12-16, 2015

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Outline

- damped NS system in the periodic domain

- Lieb–Thirring inequalities on the torus

- damped NS system in $\mathbb{R}^2$

Damped 2d Navier–Stokes system

We consider the damped and driven Navier–Stokes system

$$\begin{cases} \partial_t u + \sum_{i=1}^{2} u^i \partial_i u = \nu \Delta u - \mu u - \nabla p + g, \\ \operatorname{div} u = 0, \qquad u\big|_{t=0} = u_0. \end{cases}$$

in a periodic square domain $\mathbb{T}^2 = (0, L) \times (0, L)$. The system is studied in the limit $\nu \to 0^+$, while the drag/damping coefficient μ is arbitrary but fixed.

$$\partial_t u + B(u, u) + \nu A u = -\mu u + g, \qquad u(0) = u_0,$$

where A is the Stokes operator and $B(u, v)$ is the nonlinear term

$$\langle Au, v \rangle = (\nabla u, \nabla v), \qquad u, v \in H^1 \cap H,$$

and

$$\langle B(v, u), w \rangle = \int \sum_{i,k=1}^{2} v^k \partial_k u^i w^i \, dx =: b(v, u, w),$$

for all $u, v, w \in H^1 \cap H$, where H is $\dot{L}_2 \cap \{\operatorname{div} u = 0\}$.

H^1 estimate

Taking the scalar product of with Δu, using the identity

$$((u,\nabla)u,\Delta u)=0$$

we obtain

$$\partial_t\|\nabla u\|^2+2\nu\|\Delta u\|^2+2\mu\|\nabla u\|^2=2(\nabla g,\nabla u)\le\mu\|\nabla u\|^2+\mu^{-1}\|\nabla g\|^2.$$

Dropping the ν-term, the Gronwall inequality gives

$$\|\nabla u(t)\|^2\le\|\nabla u(0)\|^2e^{-\mu t}+\frac{1-e^{-\mu t}}{\mu^2}\|\nabla g\|^2,$$

On the attractor $u(t)\in\mathcal{A}$ letting $t\to\infty$ we have a ν-independent estimate

$$\|\nabla u(t)\|^2\le\frac{\|\nabla g\|^2}{\mu^2}.$$

Linearization and Lyapunov exponents

Variational equation

$$\partial_t U = -\nu AU - \mu U \ - B(U, u(t)) - B(u(t), U) =: \mathcal{L}(t, u_0)U, \qquad U(0) = \xi.$$

and the numbers $q(m)$:

$$q(m) \le \limsup_{t\to\infty} \sup_{u_0\in\mathcal{A}} \sup_{\{v_j\}_{j=1}^m \in H\cap H^1} \frac{1}{t}\int_0^t \sum_{j=1}^m \Big(\mathcal{L}(\tau, u_0)v_j, v_j\Big) d\tau,$$

where $\{v_j\}_{j=1}^m \in H \cap H^1$ and $(v_i, v_j) = \delta_{ij}$. For the sum we have

$$\sum_{j=1}^m \Big(\mathcal{L}(t, u_0)v_j, v_j\Big) = -\nu \sum_{j=1}^m \|\nabla v_j\|^2 - \mu m - \sum_{j=1}^m b(v_j, u(t), v_j).$$

For the last term we use the point-wise inequality ($\operatorname{div} u = 0$)

$$\Big| \sum_{k,i=1}^2 v^k \partial_k u^i v^i \Big| \le 2^{-1/2} |\nabla u| |v|^2$$

We obtain

$$\sum_{j=1}^{m}\Big(\mathcal{L}(t,u_0)v_j,v_j\Big) \le -\nu\sum_{j=1}^{m}\|\nabla v_j\|^2 - \mu m + 2^{-1/2}\int|\nabla u(t,x)|\rho_v(x)dx \le$$

$$-\nu\sum_{j=1}^{m}\|\nabla v_j\|^2 - \mu m + 2^{-1/2}\|\nabla u(t)\|\|\rho_v\|,$$

where

$$\rho_v(x) := \sum_{j=1}^{m}|v_j(x)|^2.$$

Lieb–Thirring inequalities

Theorem 1. *Let* $\{\varphi_j\}_{j=1}^m \in \dot{L}_2(\mathbb{T}^2) \cap H^1(\mathbb{T}^2)$ *and let* $(\varphi_i, \varphi_j) = \delta_{ij}$. *Then for*

$$\rho_\varphi(x) := \sum_{j=1}^m \varphi_j(x)^2$$

the following inequality holds:

$$\int_{\mathbb{T}^2} \rho_\varphi(x)^2 dx \le c_{\mathrm{LT}} \sum_{j=1}^m \|\nabla \varphi_j\|^2.$$

If $\{v_j\}_{j=1}^m \in \dot{L}_2(\mathbb{T}^2) \cap H^1(\mathbb{T}^2)$, $(v_i, v_j) = \delta_{ij}$ *and* $\operatorname{div} v_j = 0$, *then*

$$\rho_v(x) := \sum_{j=1}^m |v_j(x)|^2$$

satisfies

$$\int_{\mathbb{T}^2} \rho_v(x)^2 dx \le \vec{c}_{\mathrm{LT}} \sum_{j=1}^m \|\nabla v_j\|^2, \qquad \vec{c}_{\mathrm{LT}} \le c_{\mathrm{LT}}.$$

Finally,

$$c_{\mathrm{LT}} \le \frac{3}{2}\,.$$

Estimate for $q(m)$

We continue

$$\sum_{j=1}^{m}\Big(\mathcal{L}(t,u_0)v_j,v_j\Big) \le -\nu\sum_{j=1}^{m}\|\nabla v_j\|^2 - \mu m + 2^{-1/2}\|\nabla u(t)\|\|\rho_v\| \le$$
$$-\nu\sum_{j=1}^{m}\|\nabla v_j\|^2 - \mu m + 2^{-1/2}\|\nabla u(t)\|\left(c_{\mathsf{LT}}\sum_{j=1}^{m}\|\nabla v_j\|^2\right)^{1/2} \le$$
$$-\nu\sum_{j=1}^{m}\|\nabla v_j\|^2 - \mu m + \frac{c_{\mathsf{LT}}}{8\nu}\|\nabla u(t)\|^2 + \nu\sum_{j=1}^{m}\|\nabla v_j\|^2 =$$
$$-\mu m + \frac{c_{\mathsf{LT}}}{8\nu}\|\nabla u(t)\|^2.$$

Since

$$\|\nabla u(t)\|^2 \le \frac{\|\nabla g\|^2}{\mu^2},$$

$$q(m) \le -\mu m + \frac{c_{\mathsf{LT}}\|\nabla g\|^2}{8\nu\mu^2}.$$

We can proceed in a somewhat different way observing that

$$m = \int \rho_v(x)dx \le \|\rho_v\| L \qquad \Rightarrow \qquad \sum_{j=1}^{m} \|\nabla v_j\|^2 \ge \frac{1}{c_{\mathsf{LT}}}\|\rho_v\|^2 \ge \frac{1}{c_{\mathsf{LT}}}\frac{m^2}{L^2}.$$

Then we argue as before but single out the term $\nu/2\sum_{j=1}^m \|\nabla v_j\|^2$. We obtain

$$q(m) \le -\frac{\nu m^2}{2c_{\mathsf{LT}}L^2} + \frac{c_{\mathsf{LT}}\|\nabla g\|^2}{4\nu\mu^2}.$$

Now if for an m^* we have $q(m^*) < 0$, then both the Hausdorff dimension, and the fractal dimension of the attractor $\mathcal{A}$ satisfy

$$\dim_H \mathcal{A} \le \dim_F \mathcal{A} < m^*.$$

Theorem 2. *The fractal dimension of the attractor for the damped-driven Navier-Stokes system in the periodic domain* $\mathbb{T}^2 = (0, L) \times (0, L)$ *satisfy the estimate*

$$\dim_F \mathcal{A} \le \min\left(\frac{c_{\mathsf{LT}}}{8}\frac{\|\nabla g\|^2}{\nu\mu^3}, \frac{c_{\mathsf{LT}}}{2^{1/2}}\frac{\|\nabla g\|L}{\nu\mu}\right) \le \min\left(\frac{3}{16}\frac{\|\nabla g\|^2}{\nu\mu^3}, \left(\frac{9}{8}\right)^{1/2}\frac{\|\nabla g\|L}{\nu\mu}\right).$$

Lieb–Thirring inequalities on $\mathbb{T}^2_\alpha = (0, L/\alpha) \times (0, L)$

Theorem 3. *Let* $\{\varphi_j\}_{j=1}^m \in \dot{L}_2(\mathbb{T}^2_\alpha) \cap H^1(\mathbb{T}^2_\alpha)$ *and let* $(\varphi_i, \varphi_j) = \delta_{ij}$. *Then for*

$$\rho_\varphi(x) := \sum_{j=1}^m \varphi_j(x)^2$$

the following inequality holds:

$$\int_{\mathbb{T}^2_\alpha} \rho_\varphi(x)^2 dx \le c_{\mathrm{LT}}(\alpha) \sum_{j=1}^m \|\nabla \varphi_j\|^2.$$

If $\{v_j\}_{j=1}^m \in \dot{L}_2(\mathbb{T}^2_\alpha) \cap H^1(\mathbb{T}^2_\alpha)$, $(v_i, v_j) = \delta_{ij}$ *and* $\operatorname{div} v_j = 0$, *then*

$$\rho_v(x) := \sum_{j=1}^m |v_j(x)|^2$$

satisfies

$$\int_{\mathbb{T}^2} \rho_v(x)^2 dx \le \vec{c}_{\mathrm{LT}} \sum_{j=1}^m \|\nabla v_j\|^2, \qquad \vec{c}_{\mathrm{LT}}(\alpha) \le c_{\mathrm{LT}}(\alpha).$$

Finally,

$$c_{\mathrm{LT}}(\alpha) \le \frac{c_{\mathrm{LT}}}{\alpha} \le \frac{3}{2} \cdot \frac{1}{\alpha}.$$

Lieb–Thirring inequalities on the d-dimensional torus

$$\mathbb{T}^d_\alpha = (0, 2\pi/\alpha_1) \times \cdots \times (0, 2\pi/\alpha_{d-1}) \times (0, 2\pi),$$

where the lengths of the periods are arranged in the non-increasing order

$$\alpha_1 \le \cdots \le \alpha_{d-1} \le \alpha_d = 1,$$

Two interpolation inequalities

First inequality. Let $x \in [0, 2\pi/\alpha]_{\mathrm{per}}$ and let $\alpha > 0$. We consider the following interpolation inequality

$$\|u\|_\infty^2 \le K_1(\beta) \left(\int_0^{2\pi/\alpha} \Big(u'(x)^2 + \alpha^2 \beta u(x)^2 \Big) dx \right)^{1/2} \left(\int_0^{2\pi/\alpha} u(x)^2 dx \right)^{1/2},$$

where $u \in H^1(0, 2\pi/\alpha)_{\mathrm{per}}$, $\beta > 0$ (and no orthogonality to constants is assumed). More precisely, we are interested in the value of the sharp constant $K_1(\beta)$ in this inequality.

General method

The Green's function $G_\lambda(x,\xi)$, that is, the solution of the equation

$$\mathbb{A}(\lambda) G_\lambda(x,\xi) = \delta(x-\xi),$$

where

$$\mathbb{A}(\lambda) = -\frac{d^2}{dx^2} + \alpha^2\beta + \lambda, \qquad \lambda > 0,$$

is given by the series

$$G_\lambda(x,\xi) = \frac{\alpha}{2\pi} \sum_{k\in\mathbb{Z}} \frac{e^{ik\alpha(x-\xi)}}{\alpha^2k^2 + \alpha^2\beta + \lambda}\,.$$

On the diagonal $x = \xi$

$$G_\lambda(\xi,\xi) = \frac{\alpha}{2\pi} \sum_{k\in\mathbb{Z}} \frac{1}{\alpha^2k^2 + \alpha^2\beta + \lambda} =: g_\beta(\lambda).$$

Using the general result in (ZI) we have the following expression for $K_1(\beta)$:

$$K_1(\beta) = 2 \sup_{\lambda>0} \lambda^{1/2} g_\beta(\lambda)$$

Next, summing the series, we find that

$$g_\beta(\lambda) = \frac{1}{2\alpha}\frac{\coth(\pi(\beta+\lambda/\alpha^2)^{1/2})}{(\beta+\lambda/\alpha^2)^{1/2}}\,.$$

Since $\alpha > 0$ is fixed, we can replace the variable λ by $\alpha^2\lambda$, which finally gives

$$K_1(\beta) = \sup_{\lambda>0} \lambda^{1/2}\,\frac{\coth(\pi(\beta+\lambda)^{1/2})}{(\beta+\lambda)^{1/2}}\,.$$

Since for every fixed $\beta > 0$

$$\lim_{\lambda\to\infty} \lambda^{1/2}\,\frac{\coth(\pi(\beta+\lambda)^{1/2})}{(\beta+\lambda)^{1/2}} = 1,$$

it follows that

$$K_1(\beta) \geq 1.$$

Next, $K_1(\beta)$ is monotone non-increasing

$$\beta_1 < \beta_2 \quad \Rightarrow \quad g_{\beta_1}(\lambda) > g_{\beta_2}(\lambda) \quad \Rightarrow \quad K_1(\beta_1) \geq K_1(\beta_2),$$

and, finally,

$$K_1(\beta_0) = 1 \quad \Rightarrow \quad K_1(\beta) = 1 \quad \text{for} \quad \beta \geq \beta_0.$$

Thus, we have proved the following theorem.

Theorem 4. *The sharp constant* $K_1(\beta)$, $\beta > 0$, *in the inequality*

$$\|u\|_\infty^2 \le K_1(\beta) \left(\int_0^{2\pi/\alpha} \Big(u'(x)^2 + \alpha^2 \beta u(x)^2 \Big) dx \right)^{1/2} \left(\int_0^{2\pi/\alpha} u(x)^2 dx \right)^{1/2},$$

is given by

$$K_1(\beta) = \sup_{\lambda > 0} \lambda^{1/2} \frac{\coth(\pi(\beta+\lambda)^{1/2})}{(\beta+\lambda)^{1/2}}.$$

Furthermore, for all $\beta \ge \beta_* = 0.045\ldots$, $K_1(\beta) = 1$.

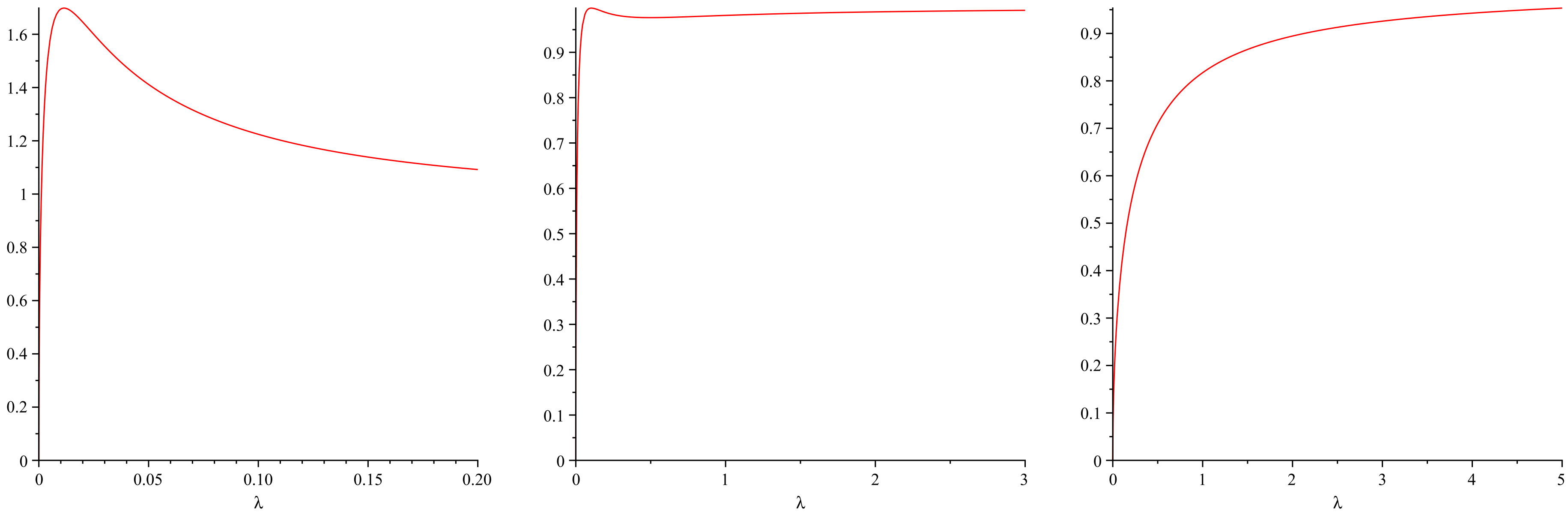

Рис. 1: Graphs of $\lambda \to \lambda^{1/2}\,\frac{\coth(\pi(\beta+\lambda)^{1/2})}{(\beta+\lambda)^{1/2}}$ for $\beta = 0.01$, $\beta_* = 0.045\ldots$, 0.5.

Second inequality. Now let $x \in [0, 2\pi]_{\text{per}}$, and the inequality we are interested in is as follows

$$\|u\|_\infty^2 \le K_2(\beta) \left(\int_0^{2\pi} \left(u'(x)^2 - \beta u(x)^2 \right) dx \right)^{1/2} \left(\int_0^{2\pi} u(x)^2 dx \right)^{1/2},$$

where $u \in \dot{H}^1(0, 2\pi)_{\text{per}}$, that is, we impose the zero mean condition:

$$\int_0^{2\pi} u(x) dx = 0,$$

and $0 \le \beta < 1$.

Theorem 5. *The sharp constant $K_2(\beta)$ is given by*

$$K_2(\beta) = \sup_{\lambda \ge 0} \lambda^{1/2} \frac{(\lambda - \beta)^{1/2} \coth(\pi(\lambda - \beta)^{1/2}) - 1/\pi}{\lambda - \beta}$$

Furthermore, for all $\beta \le \beta_{**} = 0.839\ldots$, $K_2(\beta) = 1$.

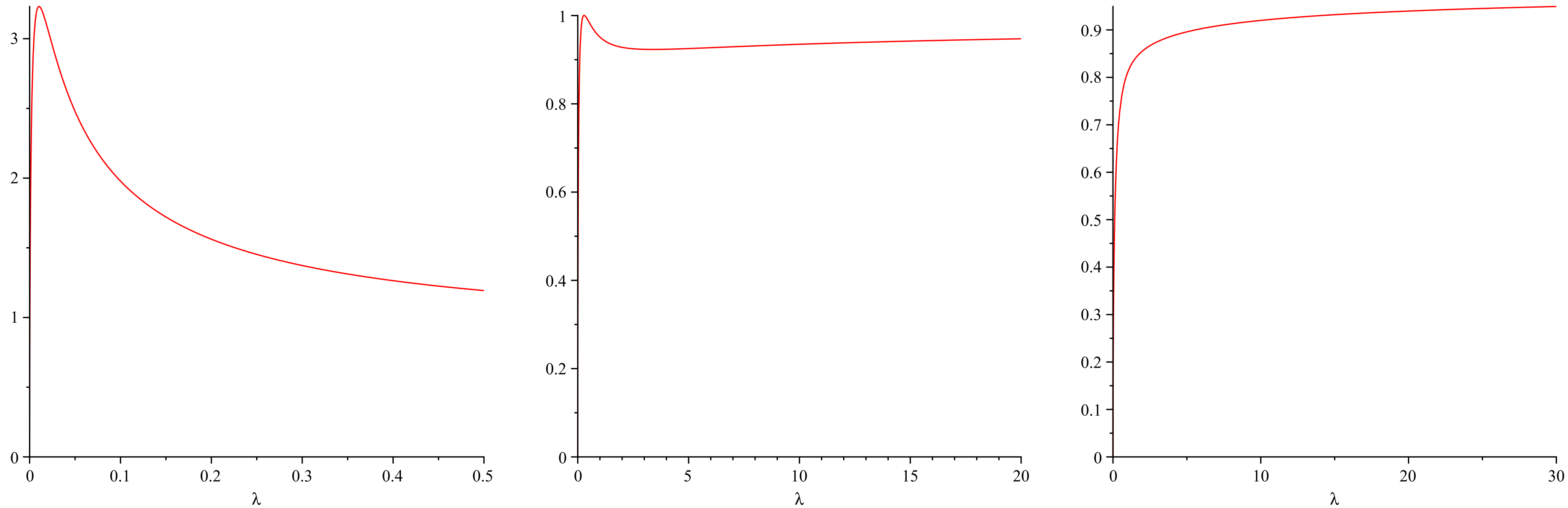

Рис. 2: Graphs of $\lambda \to \lambda^{1/2}\frac{(\lambda-\beta)^{1/2}\coth(\pi(\lambda-\beta)^{1/2})-1/\pi}{\lambda-\beta}$ for $\beta = 0.99$, $\beta_{**} = 0.839\ldots$, 0.5.

Remark 1. What is actually important in what follows are the estimates

$$\sum_{k\in\mathbb{Z}} \frac{1}{k^2+\beta+\lambda} \le K_1(\beta)\frac{\pi}{\lambda^{1/2}}, \qquad \beta > 0,$$

$$\sum_{k\in\mathbb{Z}_0} \frac{1}{k^2-\beta+\lambda} \le K_2(\beta)\frac{\pi}{\lambda^{1/2}}, \qquad \beta \in [0,1).$$

One-dimensional Sobolev inequalities for traces of matrices Let $\{\phi_n\}_{n=1}^N$ be an orthonormal family of periodic vector-functions

$$\phi_n(x) = (\phi_n(x,1), \dots, \phi_n(x,M))^T$$

defined for $x \in [0, 2\pi/\alpha]_{\text{per}}$: $(\phi_n, \phi_m) = \int_0^{2\pi/\alpha} \phi_n(x)^T \phi_m(x) dx = \delta_{nm}$.

We consider the $M \times M$ matrix $U(x,y)$

$$U(x,y) = \sum_{n=1}^N \phi_n(x)\phi_n(y)^T, \qquad [U(x,y)]_{jk} = \sum_{n=1}^N \phi_n(x,j)\phi_n(y,k).$$

Theorem 6. *Let $\phi_n(x,j) \in H^1(0, 2\pi/\alpha)_{\text{per}}$. Then for $\beta > 0$*

$$\int_0^{2\pi/\alpha} \mathrm{Tr}[U(x,x)^3] dx \le K_1(\beta)^2 \sum_{n=1}^N \sum_{j=1}^M \int_0^{2\pi/\alpha} (|\phi_n'(x,j)|^2 + \alpha^2\beta|\phi_n(x,j)|^2) dx.$$

If $\alpha = 1$ and $\int_0^{2\pi} \phi_n(x,j) = 0$ for all n and j, then for $\beta \in [0,1)$

$$\int_0^{2\pi} \mathrm{Tr}[U(x,x)^3] dx \le K_2(\beta)^2 \sum_{n=1}^N \sum_{j=1}^M \int_0^{2\pi} (|\phi_n'(x,j)|^2 - \beta|\phi_n(x,j)|^2) dx.$$

1d periodic Schrödinger operators. We consider two one-dimensional Schrödinger operators with periodic boundary conditions:

$$H_1(\beta) = -\frac{d^2}{dx^2} + \alpha^2\beta - V(x), \quad \beta > 0, \quad x \in (0, 2\pi/\alpha),$$
$$H_2(\beta) = -\frac{d^2}{dx^2} - \beta - \mathsf{P}(V(x)\cdot), \quad \beta \in [0,1), \quad x \in (0, 2\pi),$$

acting on $L_2(0, 2\pi/\alpha)$ and $\dot{L}_2(0,2\pi) = \{f \in L_2(0,2\pi),\ \int_0^{2\pi} f(x)dx = 0\}$, respectively. Here P is the orthogonal projection:

$$(\mathsf{P}\psi)(x) = \psi(x) - \frac{1}{2\pi}\int_0^{2\pi} \psi(x)dx.$$

Theorem 7. *Let V be a non-negative $M \times M$ Hermitian matrix such that $\mathrm{Tr}\, V^{3/2} \in L_1$. Then the operators $H_1(\beta)$ and $H_2(\beta)$ have discrete spectrum, and the negative eigenvalues satisfy the estimate*

$$\sum_j \lambda_j \le \frac{2}{3\cdot 3^{1/2}} K_1(\beta) \int_0^{2\pi/\alpha} \mathrm{Tr}[V(x)^{3/2}]dx,$$
$$\sum_j \mu_j \le \frac{2}{3\cdot 3^{1/2}} K_2(\beta) \int_0^{2\pi} \mathrm{Tr}[V(x)^{3/2}]dx.$$

Proof. We consider $H_2(\beta)$. Let $\{\phi_n\}_{n=1}^N$ be the orthonormal eigen-vector functions of $H_2(\beta)$ corresponding to $\{-\mu_n\}_{n=1}^N$:

$$\frac{d^2}{dx^2}\phi_n - \beta\phi_n - V\phi_n = -\mu_n\phi_n.$$

Then, using the second matrix inequality and Hölder's inequality for traces

$$\mathrm{Tr}[AB] \leq (\mathrm{Tr}([(A^*A)^{p/2}])^{1/p}(\mathrm{Tr}([(B^*B)^{p'/2}])^{1/p'}$$

and setting below $X := \int_0^{2\pi} \mathrm{Tr}[U(x,x)^3]dx$ we obtain

$$\sum_{n=1}^N \mu_n =$$

$$-\sum_{n=1}^N \sum_{j=1}^M \int_0^{2\pi} |(|\phi_n'(x,j)|^2 - \beta|\phi_n(x,j)|^2)dx + \int_0^{2\pi} \mathrm{Tr}[V(x)U(x,x)]dx \leq$$

$$\leq \left(\int_0^{2\pi} \mathrm{Tr}[V(x)^{3/2}]dx\right)^{2/3} X^{1/3} - K_2(\beta)^{-2}X.$$

Calculating the maximum with respect to X we obtain

$$\sum_j \mu_j \leq \frac{2}{3\cdot 3^{1/2}} K_2(\beta) \int_0^{2\pi} \mathrm{Tr}[V(x)^{3/2}]dx.$$

We observe that these estimates can be written in the form

$$\sum_j \lambda_j \leq \frac{\pi}{3^{1/2}} K_1(\beta) \mathsf{L}^{\mathsf{cl}}_{1,1} \int_0^{2\pi/\alpha} \mathrm{Tr}[V(x)^{3/2}]dx,$$

$$\sum_j \mu_j \leq \frac{\pi}{3^{1/2}} K_2(\beta) \mathsf{L}^{\mathsf{cl}}_{1,1} \int_0^{2\pi} \mathrm{Tr}[V(x)^{3/2}]dx,$$

where

$$\mathsf{L}^{\mathsf{cl}}_{\gamma,d} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1-|\xi|^2)_+^\gamma \, d\xi = \frac{\Gamma(\gamma+1)}{2^d \pi^{d/2} \Gamma(\gamma+d/2+1)}.$$

Corollary 1. *[Aizenmann-Lieb] Let $V \geq 0$ be a $M \times M$ Hermitian matrix, such that $\mathrm{Tr}\, V^{\gamma+1/2} \in L_1$. Then for any $\gamma \geq 1$ the negative eigenvalues of the operators $H_1(\beta)$ and $H_2(\beta)$ satisfy the inequalities*

$$\sum_j \lambda_j^\gamma \leq \frac{\pi}{3^{1/2}} K_1(\beta) \mathsf{L}^{\mathsf{cl}}_{\gamma,1} \int_0^{2\pi/\alpha} \mathrm{Tr}[V(x)^{1/2+\gamma}]dx,$$

$$\sum_j \mu_j^\gamma \leq \frac{\pi}{3^{1/2}} K_2(\beta) \mathsf{L}^{\mathsf{cl}}_{\gamma,1} \int_0^{2\pi} \mathrm{Tr}[V(x)^{1/2+\gamma}]dx.$$

LT inequalities on the torus

$$\mathbb{T}^d_\alpha = (0, 2\pi/\alpha_1) \times \cdots \times (0, 2\pi/\alpha_{d-1}) \times (0, 2\pi), \qquad \alpha_1 \le \cdots \le \alpha_{d-1} \le \alpha_d = 1.$$

We define the orthogonal projection P:

$$(\mathsf{P}\psi)(x_1, \ldots, x_d) = \psi(x_1, \ldots, x_d) - \frac{1}{2\pi} \int_0^{2\pi} \psi(x_1, \ldots, x_{d-1}, t) dt,$$

so that the resulting function has zero average with respect to shortest coordinate x_d for all $x_1, \ldots, x_{d-1}$.

We consider the Schrödinger operator

$$\mathcal{H} = -\Delta - \mathsf{P}(V(x)\,\cdot)$$

and the main idea is to write $\mathcal{H}$ as follows:

$$\mathcal{H} = \left(-\frac{d^2}{dx_1^2} + \alpha_1^2\beta_1\right) + \cdots + \left(-\frac{d^2}{dx_{d-1}^2} + \alpha_{d-1}^2\beta_1\right) + \left(-\frac{d^2}{dx_d^2} - \delta\right) - \mathsf{P}(V(x)\,\cdot),$$

where

$$\delta := \alpha_1^2\beta_1 + \cdots + \alpha_{d-1}^2\beta_{d-1} < 1.$$

Theorem 8. *The negative eigenvalues* $-\lambda_n \le 0$ *of the operator*

$$\mathcal{H} = -\Delta - \mathsf{P}(V(x)\,\cdot)$$

satisfy the bound

$$\sum_n \lambda_n^\gamma \le \mathsf{L}_{\gamma,d} \int_{\mathbb{T}^d_\alpha} V^{\gamma+d/2}(x)dx, \qquad \gamma \ge 1,$$

where

$$\mathsf{L}_{\gamma,d} \le \left(\frac{\pi}{3^{1/2}}\right)^d \prod_{j=1}^{d-1} K_1(\beta_j)\, K_2(\delta) \mathsf{L}^{\mathsf{cl}}_{\gamma,d},$$

provided that $\beta_j > 0$, $j = 1, \dots d-1$ *are chosen so small that*

$$\delta := \alpha_1^2 \beta_1 + \cdots + \alpha_{d-1}^2 \beta_{d-1} < 1.$$

If $\beta_j \ge \beta_* = 0.045\dots \quad \& \quad \delta \le \beta_{**} = 0.839\dots$, *then*

$$\mathsf{L}_{\gamma,d} \le \left(\frac{\pi}{3^{1/2}}\right)^d \mathsf{L}^{\mathsf{cl}}_{\gamma,d}. \tag{1}$$

For

$$d \le \left[\frac{\beta_{**}}{\beta_*}\right] + 1 = 19$$

this condition can be satisfied for any α, *so that* (1) *always holds.*

Proof $d = 2$. We consider the Schrödinger operator on $\mathsf{P}L_2(\mathbb{T}^2_\alpha)$:

$$\mathcal{H} = -\frac{d^2}{dx_1^2} - \frac{d^2}{dx_2^2} - \mathsf{P}(V(x_1, x_2)\cdot\,).$$

The key idea is to split $\mathcal{H}$ into the sum of two invertible operators:

$$\mathcal{H} = -\frac{d^2}{dx_1^2} + \alpha^2\beta - \frac{d^2}{dx_2^2} - \alpha^2\beta - \mathsf{P}(V(x_1, x_2)\cdot\,), \qquad \beta \in (0, 1),$$

where the operator

$$-\frac{d^2}{dx_1^2} + \alpha^2\beta$$

is invertible on $L_2(0, 2\pi/\alpha)_{\text{per}}$, since $\beta > 0$, and

$$-\frac{d^2}{dx_2^2} - \alpha^2\beta$$

is invertible on $L_2(0, 2\pi)_{\text{per}} \cap \{\int_0^{2\pi} \psi(x_2) dx_2 = 0\}$, since $\alpha^2\beta < 1$.

What we have to prove is

$$\sum_j \lambda_j \le \left(\frac{\pi}{3^{1/2}}\right)^2 \mathsf{L}^{\mathsf{cl}}_{1,2} \int_{\mathbb{T}^2_\alpha} V^2(x_1, x_2)dx_1dx_2 = \frac{\pi}{24}\int_{\mathbb{T}^2_\alpha} V^2(x_1, x_2)dx_1dx_2.$$

We use the lifting argument with respect to the dimension (LW):

$$\begin{aligned}
&\sum_j \lambda_j(\mathcal{H}) = \sum_j \lambda_j(-\partial^2_{x_1} + \alpha^2\beta - \partial^2 x_2 - \alpha^2\beta - \mathsf{P}(V(x_1, x_2)\cdot))\\
&\le \sum_j \lambda_j(-\partial^2_{x_1} + \alpha^2\beta + [-\partial^2_{x_2} - \alpha^2\beta - \mathsf{P}(V(x_1, x_2)\cdot)]_-)\\
&\le \frac{\pi}{3^{1/2}} K_1(\beta)\mathsf{L}^{\mathsf{cl}}_{1,1} \int_0^{2\pi/\alpha} \mathsf{Tr}\left[-\partial^2_{x_2} - \alpha^2\beta - \mathsf{P}(V(x_1, x_2)\cdot)\right]^{3/2}_- dx_1\\
&= \frac{\pi}{3^{1/2}} K_1(\beta)\mathsf{L}^{\mathsf{cl}}_{1,1}\frac{\pi}{3^{1/2}} K_2(\alpha^2\beta)\mathsf{L}^{\mathsf{cl}}_{3/2,1}\int_0^{2\pi/\alpha}\int_0^{2\pi} V^2(x_1, x_2)dx_1dx_2,
\end{aligned}$$

where we have used one dimensional estimate in the matrix case with $L = 2\pi/\alpha$, $\gamma = 1$ and then 1d estimate in the scalar case with $L = 2\pi$, $\alpha = 1$, and $\beta = \alpha^2\beta$. If we take $\beta = 1/2$, then $K_1(\beta) = 1$. Furthermore, since $\alpha \le 1$, we have $\alpha^2\beta \le 1/2$, and therefore $K_2(\alpha^2\beta) = 1$. It remains to notice that

$$\mathsf{L}^{\mathsf{cl}}_{1,1}\mathsf{L}^{\mathsf{cl}}_{3/2,1} = \mathsf{L}^{\mathsf{cl}}_{1,2}.$$

Damped NS on the elongated torus $\mathbb{T}^2_\alpha$

Let us consider the damped NS system on the large elongated torus $\mathbb{T}^2_\alpha = (0, L/\alpha)\times(0, L)$, where $\alpha \le 1$. As before we assume that both scalar and vector functions have mean value zero over $\mathbb{T}^2_\alpha$, and we decompose the phase space

$$\dot L_2(\mathbb{T}^2_\alpha) = \Pi L_2(\mathbb{T}^2_\alpha) = \{f(x_1, x_2) - \frac{\alpha}{L^2}\int_0^{L/\alpha}\int_0^L f(x_1, x_2)dx_1dx_2\}$$

into the orthogonal sum

$$\dot L_2(\mathbb{T}^2_\alpha) = \mathsf{P}\dot L_2(\mathbb{T}^2_\alpha) \oplus \mathsf{Q}\dot L_2(\mathbb{T}^2_\alpha),$$

where the orthogonal projection in the 2d case P is as as before:

$$(\mathsf{P}\psi)(x_1, x_2) = \psi(x_1, x_2) - \frac{1}{L}\int_0^L \psi(x_1, t)dt,$$

and the projection Q

$$(\mathsf{Q}\psi)(x_1) = \frac{1}{L}\int_0^L \psi(x_1, t)dt$$

maps $\dot L_2(\mathbb{T}^2_\alpha)$ onto $\dot L_2(0, L/\alpha)$.

Theorem 9. *Let* $\{v_j\}_{j=1}^m \in \mathsf{P}\dot{L}_2(\mathsf{T}^2_\alpha)$ *be orthonormal and* $\operatorname{div} v_j = 0$. *Then*

$$\rho_{\mathsf{P}v}(x) = \sum_{j=1}^m |v_j(x)|^2$$

satisfies

$$\int_{\mathsf{T}^2_\alpha} \rho_{\mathsf{P}v}(x)^2 dx \le c_{\mathsf{P}} \sum_{j=1}^m \|\nabla v_j\|^2, \qquad c_{\mathsf{P}} \le \frac{\pi}{6} = \frac{\pi}{24} \cdot 4.$$

Accordingly, if $\{w_j\}_{j=1}^m \in \mathsf{Q}\dot{L}_2(\mathsf{T}^2_\alpha)$ *is orthonormal and* $\operatorname{div} w_j = 0$, *then*

$$\rho_{\mathsf{Q}w}(x) = \sum_{j=1}^m |w_j(x)|^2,$$

satisfies (essentially, a one-dimensional inequality)

$$\int_{\mathsf{T}^2_\alpha} \rho_{\mathsf{Q}w}(x)^2 dx \le \frac{c_{\mathsf{Q}}}{L} \sum_{j=1}^m \|(-d^2/dx_1^2)^{1/4} w_j\|^2 \le \frac{c_{\mathsf{Q}}}{L} \sum_{j=1}^m \|\nabla w_j\|,$$

where

$$c_{\mathsf{Q}} \le 6.$$

Theorem 10. *The damped Navier–Stokes system*

$$\begin{cases} \partial_t u + \sum_{i=1}^2 u^i \partial_i u = \nu \Delta u - \mu u - \nabla p + g, \\ \operatorname{div} u = 0, \quad u\big|_{t=0} = u_0. \end{cases}$$

on the elongated torus $\mathsf{T}^2_\alpha = (0, L/\alpha) \times (0, L)$ *has the attractor* $\mathcal{A}$ *and*

$$\dim_F \mathcal{A} \le \left(\frac{c_{\mathsf{P}}}{2} + (c_{\mathsf{P}} c_{\mathsf{Q}})^{1/2}\right) \frac{\|\nabla g\|^2}{\nu \mu^2} \le \left(\frac{\pi}{12} + \pi^{1/2}\right) \frac{\|\nabla g\|^2}{\nu \mu^3}$$

for all sufficiently small $\nu \le 8\mu L^2$.

$$u = \mathsf{P}u + \mathsf{Q}u, \qquad v_j = \mathsf{P}v_j + \mathsf{Q}v_j$$

Remark 2. The α-dependence is (implicitly) contained only in $\|\nabla g\| = \|\nabla g\|_{L_2(\mathbb{T}^2_\alpha)}$

The upper bound is sharp as both $\nu \to 0$ and $\alpha \to 0$. The right-hand side

$$g = g_s = \begin{cases} g_1 = c_1 \nu^2 s^3 \sin s x_2, \\ g_2 = 0, \end{cases}$$

where c_1 is an absolute constant and $\mathbb{T}^2_\alpha = (0, 2\pi/\alpha) \times (0, 2\pi)$, produces the stationary solution with unstable manifold of dimension

$$d = c_2 \frac{s^2}{\alpha},$$

where $s \gg 1$. Setting $s := \left(\frac{\mu}{\nu}\right)^{1/2}$ we find that $\dim \mathcal{A} \geq d = c_2 \frac{\mu}{\alpha\nu}$. Since g is independent of x_1, it follows that

$$\|\nabla g_s\|^2 = c_3 \frac{\nu^4 s^8}{\alpha} = c_3 \frac{\mu^4}{\alpha},$$

so that for $g = g_s$ the dimensionless number $\frac{\|\nabla g\|^2}{\nu \mu^3}$ becomes

$$\frac{\|\nabla g\|^2}{\nu \mu^3} = c_3 \frac{\mu}{\alpha\nu} \qquad \Rightarrow \qquad \dim \mathcal{A}_s \geq \frac{c_2}{c_3} \frac{\|\nabla g\|^2}{\nu \mu^3}.$$

The dimensionless number

$$\frac{\|\nabla g\|^2}{\nu\mu^3}$$

depends on the domain only via $\|\nabla g\|^2$.

Therefore we can expect that the above upper bound holds for $x \in \mathbb{R}^2$ and

$$\|\nabla g\| < \infty$$

in the space of finite energy solutions $L_2(\mathbb{R}^2)$.

This is indeed the case, furthermore, we can include the forcing term g in the family of homogeneous Sobolev spaces

$$\dot{H}^s(\mathbb{R}^2) = (-\Delta)^{-s/2} L^2(\mathbb{R}^2)$$

for $s \in \mathbb{R}$. The norm in this space is given by

$$\|u\|^2_{\dot{H}^s} := \int_{\mathbb{R}^2} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi,$$

where

$$\hat{u}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}^2} u(x) e^{-i\xi x} \, dx$$

is the Fourier transform of u.

$$x \in \mathbb{R}^2$$

Theorem 11. *Let $g \in \dot{H}^s(\mathbb{R}^2)$ for some $s \in [-1,1]$. Then, the fractal dimension of the attractor $\mathcal{A}$ satisfies the following estimate:*

$$\dim_F \mathcal{A} \leq \frac{1}{64 \cdot 3^{1/2}}(1-s^2) \cdot \left(\frac{1+|s|}{1-|s|}\right)^{|s|} \frac{1}{\mu^2\nu^2} \left(\frac{\nu}{\mu}\right)^s \|g\|^2_{\dot{H}^s}.$$

If $g \in \dot{H}^{-1} \cap \dot{H}^1$, then the rate of growth of this estimate with respect to ν as $\nu \to 0$ is the smallest when $s = 1$. In this case this theorem gives the following result.

Corollary 2. *Suppose that $g \in \dot{H}^1$. Then the fractal dimension of the attractor $\mathcal{A}$ satisfies*

$$\dim_F \mathcal{A} \leq \frac{1}{16 \cdot 3^{1/2}} \frac{\|\nabla g\|^2}{\nu\mu^3}.$$

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Thank you for your attention