Normalization of Equation with Two Delays of Different Order

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Infinite-dimensional dynamics, dissipative systems, and attractors
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Outline

1. Problem definition
2. Linear analysis
3. Quainormal forms
4. Case of small multiplier at term with largest delay

Problem Definition

Equation with two delays

$$\dot{x}+x = ax(t-T_1)+bx(t-T_2)+f(x, x(t-T_1), x(t-T_2)), \quad T_1, T_2 > 0$$

1. T_1 is some fixed value; T_2 is large.
2. T_1 and T_2 are large and have same order.
3. T_1 and T_2 are large and $T_2 \gg T_1$.

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Main assumtions

$$T_1 = \frac{1}{\varepsilon}, \qquad T_2 = \frac{1}{c\varepsilon^{1+\alpha}}, \qquad c > 0, \alpha > 0,$$

$$0 < \varepsilon \ll 1.$$

After some renorm

$$\varepsilon\dot{x} + x = ax(t-1) + bx\big(t - \frac{1}{c\varepsilon^{\alpha}}\big) + f\left(x, x(t-1), x\big(t - \frac{1}{c\varepsilon^{\alpha}}\big)\right)$$

Goal: study local dynamics in neighbourhood of zero equilibrium for sufficiently small ε.

Linear Analysis

Linearized equation

$$\varepsilon\dot{x} + x = ax(t-1) + bx\left(t - \frac{1}{c\varepsilon^{\alpha}}\right)$$

Characteristic equation

Use standart ansatz $x = \exp(\lambda t) \Longrightarrow$

$$\varepsilon\lambda + 1 = a\exp(-\lambda) + b\exp\left(-\frac{\lambda}{c\varepsilon^{\alpha}}\right)$$

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Linear Analysis. Results

Lemma 1 (Stability)

If $|a| + |b| < 1$ then exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$ all roots of characteristc equation has neative real parts. Zero equilibrium is stable.

Lemma 2 (Unstability)

if $|a| + |b| > 1$ then exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$ there exists root of characteristic equation with positive real part. Zero equilibrium state is unstable.

Lemma 3 (Critical case)

if $|a| + |b| = 1$ then exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$ all roots of characteristic equation satisfy $Re\lambda < M(\varepsilon) = o(1)$. And there are infinite number of roots λ_k $(k \in \mathbb{Z})$ such that $Re\lambda_k \to 0$ as $\varepsilon \to 0$.

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Method

Let a_0 and b_0 — critical values (i.e. $|a_0| + |b_0| = 1$).

$$a = a_0 + \mu a_1, \qquad b = b_0 + \mu b_1, \quad 0 < \mu \ll 1$$

$$\mu = \varepsilon^{\beta}, \qquad \beta > 0.$$

β is important:

- $\beta = 2\min(1, \alpha)$
- $\beta < 2\min(1, \alpha)$

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Case $b_0 \neq 0$

Let $a_0 \geq 0,\ b_0 > 0\ (a_0 + b_0 = 1),\ \beta = 2\min(1, \alpha)$

Critical roots

$$\lambda_{nk} = 2\pi ik + \varepsilon^{\alpha}(2\pi n + k\theta)ic - \varepsilon^{\alpha}\frac{c}{b_0}(2\pi k\varepsilon + a_0 c(2\pi n + k\theta)\varepsilon^{\alpha})i + \dots$$

$$-\varepsilon^{3\alpha}\frac{a_0 c^3}{2b_0}(2\pi n + k\theta)^2 - \varepsilon^{\alpha}\frac{a_0 c^3}{2b_0^3}(2\pi k\varepsilon + a_0 c(2\pi n + k\theta)\varepsilon^{\alpha})^2 + \varepsilon^{\alpha+\beta}\frac{c}{b_0}(a_1 + b_1))$$

$$+ \dots$$

$\theta = 2\pi\theta_1(\varepsilon)$, $\theta_1 \in [0, 1)$: $\frac{1}{c\varepsilon^{\alpha}} + \theta_1$ is integer.

$\alpha > 1$, $\beta = 2\min(1, \alpha) = 2$

Substitution

$$x(t,\varepsilon) = \varepsilon^2 u(\tau, s) + \varepsilon^4 u_2(\tau, s) + \dots$$

Here $\tau = \varepsilon^2 t$, $s = (1 + \varepsilon c\theta_1 - \varepsilon^\alpha c b_0^{-1}(\varepsilon + \varepsilon^\alpha a_0 c\theta_1) + \dots)t$ and $u_2(\tau, s)$ is a periodic by s function.

Quasinormal form

$$\frac{\partial u}{\partial \tau} = \frac{a_0 c^3}{2b_0^2}\frac{\partial^2 u}{\partial s^2} + (a_1 + b_1)u + du^2, \qquad u(\tau, s) = u(\tau, s+1)$$

Theorem

Let $u(\tau, s)$ be solution of QNF, bounded as well as its derivatives. Then function

$$x(t) = \varepsilon^2 u(\varepsilon^2 t, (1 + o(1))t)$$

satisfy initial equation with asymptotically small ($o(\varepsilon^2)$) residual.

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$$x(t, \varepsilon) = \varepsilon^{2\alpha} u(\tau, s, r) + \varepsilon^{4\alpha} u_2(\tau, s, r) + \dots$$

Here $\tau = \varepsilon^{2\alpha} t$, $s = (1 + \varepsilon c\theta_1 - \varepsilon^{\alpha} c b_0^{-1}(\varepsilon + \varepsilon^{\alpha} a_0 c\theta_1) + \dots)t$
$r = \varepsilon^{\alpha}(c - \varepsilon^{\alpha} a_0 c^2 b_0^{-1})t$ and $u_2(\tau, s, r)$ is a periodic by s and by r function.

Quasinormal form

$$\frac{\partial u}{\partial \tau} = \frac{a_0 c^3}{2 b_0}\left(1 + \frac{a_0^2 c^2}{b_0^2}\right)\left(\frac{\partial}{\partial r} + \theta_1 \frac{\partial}{\partial s}\right)^2 u + (a_1 + b_1)u + du^2$$

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When $\varepsilon \to 0$ function θ_1 take any value from $[0, 1)$ infinitely many times. $\varepsilon_m \to 0$ as $m \to \infty$ such that $\theta_1 = Q$

Theorem

Let for $\theta_1 = Q$ QNF has bounded with its derivatives solution $u(\tau, s, r)$. Then

$$x(t) = \varepsilon^{2\alpha} u(\varepsilon^{2\alpha} t, (1 + o(1))t, \varepsilon^{\alpha}(1 + o(1))t) + o(\varepsilon^{2\alpha})$$

satisfy initial equation with asymptotically small residual for $\varepsilon = \varepsilon_m$.

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$\alpha = 1,\ \beta = 2\min(1,\alpha) = 2$

Substitution

$$x(t,\varepsilon) = \varepsilon^2 u(\tau, s, r) + \varepsilon^4 u_2(\tau, s, r) + \dots$$

Here $\tau = \varepsilon^2 t,\ s = (1 + \varepsilon c\theta_1 - \varepsilon^2 c b_0^{-1}(1 + a_0 c\theta_1) + \dots)t$
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Case $\beta < \min(\alpha, 1)$

Let $a_0 \geq 0,\ b_0 > 0,\ \beta < 2\min(1, \alpha)$

Critical roots

$$\lambda_{nk} = (z\varepsilon^{-\gamma} + \Theta_z)ki + \varepsilon^{\alpha}(\omega\varepsilon^{-\delta} + \Theta_\omega)ni + \ldots -$$

$$-\varepsilon^{3\alpha-2\delta}\frac{a_0c^3}{2b_0}\omega^2 - \varepsilon^{\alpha}\frac{a_0c^3}{2b_0^3}(zk\varepsilon^{1-\gamma} + a_0c\omega n\varepsilon^{\alpha-\delta})^2 + \varepsilon^{\alpha+\beta}\frac{c}{b_0}(a_1 + b_1))$$

$$+\ldots$$

z and ω are arbitrary ($z\omega \neq 0$)
$\Theta_z \in [0, 2\pi)$: $\frac{z}{\varepsilon^\gamma} + \Theta_z$ is multiply 2π
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Substitution

$$x(t, \varepsilon) = \varepsilon^{\beta} u(\tau, s) + \varepsilon^{2\beta} u_2(\tau, s) + \dots$$

Here $\tau = \varepsilon^{\beta} t$, $s = (z\varepsilon^{\beta/2-1} + \Theta_1 + \dots)t$ and $u_2(\tau, s)$ is a periodic by s.

Quasinormal form

$$\frac{\partial u}{\partial \tau} = z^2 \frac{a_0 c^3}{2b_0^2} \frac{\partial^2 u}{\partial s^2} + (a_1 + b_1)u + du^2, \qquad u(\tau, s) = u(\tau, s+1)$$

Theorem

Let $u(\tau, s)$ be solution of QNF for some $z \neq 0$, bounded as well as its derivatives. Then

$$x(t) = \varepsilon^{\beta} u(\varepsilon^{\beta} t, (\frac{z}{\varepsilon^{1-\beta/2}} + \Theta_z)(1 + o(1))t) + o(\varepsilon^{\beta})$$

satisfy initial equation with asymptotically small residual.

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$\alpha < 1$, $\beta < \min(\alpha, 1) = 2\alpha$

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$\omega \neq 0$ is arbitrary.

$$u(\tau, r) = u(\tau, r+1)$$

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$$u(\tau, s, r) = u(\tau, s+1, r) = u(\tau, s, r+1)$$

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Another signs of a_0 and b_0

Let $a_0 < 0$ or $b_0 < 0$

Quasinormal forms – parabolic equations with one or two spatial variables (depends on α, with arbitrary parameters (depends on β) and qubic nonlinearity.

$\alpha = 1$ $(\beta < 2)$. $a_0, b_0 < 0$

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$$u(\tau, s, r) = -u(\tau, s+1, r) = -u(\tau, s, r+1)$$

$$x(t) = \varepsilon^{\beta/2} u(\varepsilon^\beta t, (\frac{z}{\varepsilon^{1-\beta/2}} + \Theta_z)(1+o(1))t, \varepsilon(\frac{\omega}{\varepsilon^{1-\beta/2}} + \Theta_\omega)(1+o(1))t) + \\ + o(\varepsilon^{\beta/2})$$

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Case $b_0 = 0$

$$\varepsilon\dot{x} + x = (a_0 + \varepsilon^\beta a_1)x(t-1) + \varepsilon^\beta b_1 bx(t - \frac{1}{c\varepsilon^\alpha}) + f(x, x(t-1), x(t - \frac{1}{c\varepsilon^\alpha})).$$

Critical cases

$$a_0 = \pm 1$$

Case $b_0 = 0$

$$\varepsilon\dot{x} + x = (a_0 + \varepsilon^{\beta} a_1) x(t-1) + \varepsilon^{\beta} b_1 b x(t - \frac{1}{c\varepsilon^{\alpha}}) + f(x, x(t-1), x(t - \frac{1}{c\varepsilon^{\alpha}})).$$

Critical cases

$$a_0 = \pm 1$$

$a_0 = 1,\ b_0 = 0$

Critical roots

$$\lambda_k = (z\varepsilon^{\beta/2-1} + \Theta + o(1))ki + \varepsilon^{2\beta}\lambda_{2k} + o(\varepsilon^{2\beta}).$$

λ_{2k} is defined by equation

$$\lambda_{2k} = -2z^2k^2 + a_1 + b_1 \exp(-c\varepsilon^{\beta-\alpha}\lambda_{2k} - 2\pi ik(1-\theta_2)c).$$

Here $\theta_2 = \theta_2(\varepsilon) \in [0,1)$ such that $(\varepsilon^{-\alpha} - \varepsilon^{-1}) + \theta_2$ is integer.

$$\dot{y}_k = (-2z^2k^2 + a_1)y_k + b_1 \exp(-2\pi ik(1-\theta_2)c)y_k(t - c\varepsilon^{\beta-\alpha}).$$

Quasinormal form

$$\frac{\partial u}{\partial \tau} = \frac{z^2}{2}\frac{\partial^2 u}{\partial r^2} + a_1 u + b_1 u(\tau - c\varepsilon^{\beta-\alpha}, r + \theta_2) + du^2$$

$$u(\tau, r) = u(\tau, r+1)$$

$a_0 = 1,\ b_0 = 0$. Result

Quasinormal form

$$\frac{\partial u}{\partial \tau} = \frac{z^2}{2}\frac{\partial^2 u}{\partial r^2} + a_1 u + b_1 u(\tau - c\varepsilon^{\beta-\alpha}, r + \theta_2) + du^2$$

$$u(\tau, r) = u(\tau, r+1)$$

Theorem

Let $u(\tau, r)$ be bounded solution of QNF. Then

$$x(t) = \varepsilon^{\beta} u(\varepsilon^{\beta} t, (z\varepsilon^{\beta/2-1} + \Theta + o(1))t)$$

satisfy initial equation with asimptotically small residual.

$a_0 = -1,\ b_0 = 0$

Quasinormal form. $a_0 = -1$

$$\frac{\partial u}{\partial \tau} = \frac{z^2}{2}\frac{\partial^2 u}{\partial r^2} + a_1 u + b_1 u(\tau - c\varepsilon^{\alpha-\beta}, r + \theta_1) + d_1 u^3$$

$$u(\tau, r) = -u(\tau, r+1)$$

Conclusion

- Contructed QNF can be solved numerically, instead of principal system
- z, ω – arbitrary parameters $\Rightarrow$ multistability
- Operators $z^2 \frac{\partial^2}{\partial s^2}$ can be replaced by $\left(\sum_{j=1}^{q} z_j \frac{\partial}{\partial s_j} \right)^2$
- QNF depends on $\varepsilon \Rightarrow$ infinite process of straight and backward bifurcarions as $\varepsilon \to 0$. "Breathing".

Thank you!