Reduced ODE Systems Governing Coarsening Dynamics of Dewetting Liquid Films

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Joint work with Barbara Wagner (TU Berlin), Barbara Niethammer (Uni Bonn) and Lutz Recke (HU Berlin)

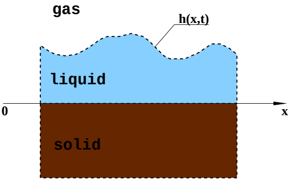
Infinite-dimensional dynamics, dissipative systems, and attractors
Nizniy Novgorod, 15 July 2014

Outline of the talk

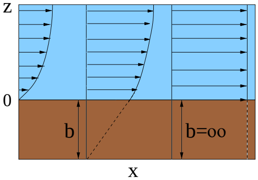
- Thin-film equations: weak- and strong-slip models.
- Energy and entropy equalities.
- Existence of weak solutions and convergence to limiting cases.

- Coarsening dynamics of drops in thin liquid films.
- Center-manifold and formal matched asymptotics approaches.
- Reduced ODE models describing coarsening dynamics.
- Coarsening laws for the exact collision-absorption model.

Physical Model



Geometric sketch of 2D liquid film



Three flow types for different slip lengths

Driving physical effects:

- Surface tension
- Intermolecular interactions with solid substrate: destabilizing van der Waals and stabilizing Born repulsion terms

$$\Pi_\varepsilon(h) = \frac{\varepsilon^2}{h^3} - \frac{\varepsilon^3}{h^4}, \quad \varepsilon \ll 1$$

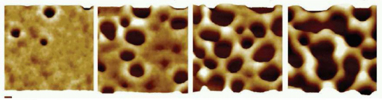
- Slippage $\boxed{b := u/u_Z}$

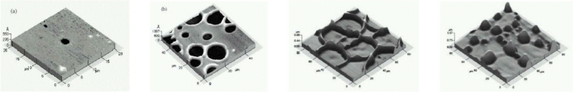
Dewetting stages

Nanometric viscous polymer fluid on SiO substrate:

Initial Instabilities ➔ **Quasi-stationary droplets** ➔

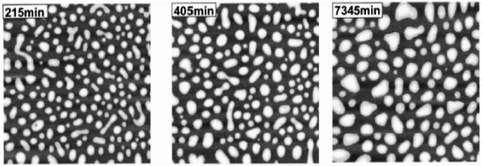
➔ **Coarsening process (collision vs. collapse)**

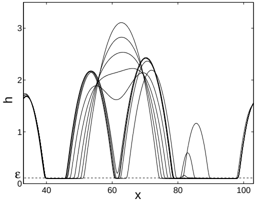




J. Becker et al. ’02, P. F. Green et. al. ’01

Last stage: Long-time Coarsening Process





experiments by Limary et al. '02, **numerics in 2D case**

Problems to consider here are:

- Coarsening mechanisms: collapse (similar to Ostwald Ripening in binary alloys) and collision of droplets (new effect)
- Coarsening rates and their dependence on physical parameters

Hierarchy of Mathematical Models

Navier-Stokes equations+conservation of mass

⬇

Lubrication equations for different slip ranges

⬇

Reduced ODE models describing coarsening dynamics of droplets

⬇

An exactly solvable collision/absorption model

Lubrication Models

Weak-slip	Strong-slip
$h_t = -\left(M(h)\left[\sigma h_{xx} - \Pi_\varepsilon(h)\right]_x\right)_x$	$\mathrm{Re}\left((hu)_t + (hu^2)_x\right) = \nu(hu_x)_x + h(\sigma h_{xx} - \Pi_\varepsilon(h))_x - \frac{u}{\beta}$ $\partial_t h = -\left(hu\right)_x$

- Conservation of mass $\int_{-L}^{L} h(x,t) = h_c = \text{const for all } t \geq 0$
- Different scalings for slip length $b \ll \beta_I \ll \beta$
- Mobility term $M(h) = h^3 + b\,h^2$
- Pressure $P(h) = -\sigma h_{xx} + \Pi_\varepsilon(h)$
- Limiting cases:
  - No-slip case $b = 0, \; M(h) = h^3$
  - Navier-slip case $\beta = \beta_I, \; M(h) = h^2$
  - Suspended free films $\beta = \infty$

Summary on the strong slip and free film models

- Derivation of the model (Münch et. al '06, Erneux and Davis '93)
- Weak solutions and their convergence to the classical solutions of the intermediate-slip equation as $\beta \to 0$. (K., Laurençot and Niethammer '11)
- Coarsening dynamics of metastable droplets. Coarsening rates for the weak-slip regime $t^{-2/5}$. Migration direction change. Conjecture of other coarsening slopes due to dominating migration. (K. and Wagner '10, Otto et. al '06, Glasner et. al '09).

Equations related to the strong-slip model

1-D Korteweg and viscous shallow-water equations on a bounded domain

$$\partial_t(hu) + \partial_x(hu^2) = \partial_x(\nu(h)\partial_x u) + \sigma h \partial_x^3 h - \partial_x P(h)$$
$$\partial_t h = -\partial_x (hu) \,.$$

- Solonnikov ‘76 - strong solutions in the case $\sigma = 0,\ \nu(h) = \nu = \text{const}$ for $d \geq 1$.
- Mellet and Vasseur ’08 and H.-L. Li et al. ’08- strong solutions in the case $\sigma = 0,\ \nu(h) = \nu h^k$ and $P(h) = h^\gamma$.
- Bresch and Desjardins ’04- weak solutions in the case $\sigma \geq 0,\ \mu(h) = \nu h$ for $d \geq 1$.

Difference to the strong-slip model in the singular pressure term and additional slip term.

Energy and entropy equalities: weak-slip model

- Energy equality:
$$\frac{dE}{dt} = -\int M(h)|\partial_x \pi|^2 \, dx,$$
where energy
$$E(h) := \int U(h) + \frac{\sigma(\partial_x h)^2}{2} \, dx$$
with $U(h) := \int_h^\infty \Pi(\tau) \, d\tau$.

- Entropy equality with entropy $G_n''(h) := 1/h^n$ (Bernis and Friedman '00):
$$\frac{d}{dt}\int G_n(h) \, dx = -\sigma \int |\partial_{xx} h|^2 \, dx - \int \Pi'(h)|\partial_x h|^2 \, dx.$$

Energy and entropy equalities: strong-slip model

- Energy equality:

$$\frac{dE}{dt} = -4\int_0^1 \nu h|\partial_x u|^2\,dx - \int_0^1 \frac{u^2}{\beta}\,dx,$$

where energy

$$E(h) := \int_0^1 \left[\mathrm{Re}\, h\frac{u^2}{2} + U(h) + \sigma\frac{|\partial_x h|^2}{2}\right]\,dx.$$

- BD-entropy equality with entropy $G_2(h) := \log(h)$:

$$\frac{d}{dt}\int_0^1 \left[\frac{1}{2}h(\mathrm{Re}\, u + \nu\partial_x G_2(h))^2 - \frac{\nu}{\beta}G_2(h) + \mathrm{Re}\left(\sigma\frac{|\partial_x h|^2}{2} + U(h)\right)\right]$$
$$= -\mathrm{Re}\int_0^1 \frac{u^2}{\beta}\,dx - 4\sigma\nu\int_0^1 |\partial_{xx}h|^2\,dx - \nu\int_0^1 \Pi'(h)|\partial_x h|^2\,dx.$$

.

A priori estimates

Proposition

For fixed positive $\sigma, \mathrm{Re}, \beta, T$ *there exists* $C_0 > 1$ *depending on* T, α, ν, Re, σ, β, *and* u_0, h_0 *such that the following terms are bounded by* C_0 *in respective norms*

$$\begin{aligned} \sqrt{h},\ \partial_x\sqrt{h},\ h^{-3/2},\ \partial_x h,\ \sqrt{\mathrm{Re}}\sqrt{h}u \quad &\in \quad L^\infty(0,T;L^2(0,1)), \\ \partial_x(h^{-3/2}),\ \partial_{xx}h,\ \sqrt{h}\partial_x u,\ \frac{u}{\sqrt{\beta}} \quad &\in \quad L^2((0,1)\times(0,T)), \end{aligned}$$

and

$$C_0^{-1} \le h(x,t) \le C_0$$

for all $x \in (0,1)$ *and* $t \in (0,T)$. *The constant* C_0 *tends to* ∞ *as* $\sigma \to 0$.

Sketch of the proof: All estimates follow from the energy equality except one for $\partial_{xx}h$.

Global weak solutions: weak-slip model

Theorem (Bernis and Friedman '90)

Let $\Pi(h) \equiv 0$ *and* $M(h) = h^n$ *with* $n > 1$ *then under some regularity conditions on* $h_0 \geq 0$ *there exists* $h \geq 0$ *a week solution on* $(0,1) \times [0,\infty)$*:*

$$h_x \in L^2(0,T;H_0^1(0,1)) \text{ for all } T > 0 \text{ and}$$

$$\int_0^\infty \int_0^1 h\partial_t\psi \, dx \, dt + \int_0^1 h_0\psi(.,0) \, dx = \sigma \int_0^\infty \int_0^1 \partial_{xx}h\partial_x(M(h)\partial_x\psi) \, dx \, dt$$

$\forall \psi \in C_0^\infty((0,1) \times [0,\infty))$. *For* $n \geq 4$ h *is a unique positive smooth solution.*

Theorem (Bertozzi et al. '01, Bertozzi and Pugh '98)

Let

$$\Pi(h) = \frac{1}{h^3} - \frac{\alpha}{h^4}, \ \alpha > 0$$

and $M(h) = h^n$ *with* $n > 1$ *then under some regularity conditions on* $h_0 > 0$ *there exists a unique smooth positive solution* h *on* $(0,1) \times [0,\infty)$.

Global weak solutions: strong-slip model

Theorem (K., Laurençot and Niethammer '11)

For any nonnegative $\sigma, \mathrm{Re}, \beta$ *and* $h_0 > 0$ *there exists a global weak solution* $(h,\, u)$ *having the regularity properties stated in a priori estimates and satisfying*

$$\int_0^\infty \int_0^1 h\partial_t\psi \, dxdt + \int_0^1 h_0\psi(\cdot,0)\, dx = -\int_0^\infty \int_0^1 hu\partial_x\psi \, dxdt,$$

$$\mathrm{Re} \int_0^\infty \int_0^1 hu\partial_t\phi \, dxdt + \mathrm{Re} \int_0^1 h_0u_0\phi(\cdot,0)\, dx + \mathrm{Re} \int_0^\infty \int_0^1 hu^2\partial_x\phi \, dx\, dt$$

$$-\nu \int_0^\infty \int_0^1 h\partial_x u\partial_x\phi \, dx\, dt - \sigma \int_0^\infty \int_0^1 \partial_x h\partial_{xx}h\phi \, dx\, dt$$

$$-\sigma \int_0^\infty \int_0^1 h\partial_{xx}h\partial_x\phi \, dx\, dt + \int_0^\infty \int_0^1 \Pi_1(h)\partial_x\phi \, dxdt - \frac{1}{\beta}\int_0^\infty \int_0^1 u\phi \, dxdt = 0$$

for all $\psi \in C_0^\infty([0,1]\times[0,\infty))$ *and* $\phi \in C_0^\infty((0,1)\times[0,\infty))$*, where*

$$\Pi_1(h) := -\int_h^\infty \tau\Pi'(\tau)\, d\tau.$$

Global weak solutions: compactness

Sketch of the proof:

- $(\partial_t h_{\varepsilon_n})$ is bdd in $L^\infty(0,T;H^{-1}(0,1))$
- (h_{ε_n}) is bdd in $L^\infty(0,T;H^1(0,1))$ and $L^2(0,T;H^2(0,1))$
- $H^1(0,1) \hookrightarrow C([0,1]) \hookrightarrow H^{-1}(0,1)$ and Simon '87 imply

$$\begin{aligned} h_{\varepsilon_n} &\to h \text{ in } L^2(0,T;W^{1,p}(0,1)) \cap C([0,1]\times[0,T]), \\ \partial_t h_{\varepsilon_n} &\overset{\star}{\rightharpoonup} \partial_t h \text{ in } L^\infty(0,T;H^{-1}(0,1)). \end{aligned}$$

- Hence by uniform low bound

$$h_{\varepsilon_n}^{-1} \to h^{-1} \text{ in } C([0,1]\times[0,T]).$$

- Next, using momentum equation and a priori estimates

$(\partial_t(h_{\varepsilon_n} u_{\varepsilon_n}))$ and $(h_{\varepsilon_n} u_{\varepsilon_n})$are bdd in $L^2(0,T;H^{-3}(0,1))$ and $L^2(0,T;H^1(0,1))$.

- Simon '87 ensures that $(h_{\varepsilon_n} u_{\varepsilon_n})$ is compact in $L^2((0,1)\times(0,T))$:

$$\partial_x u_{\varepsilon_n} \rightharpoonup \partial_x u \text{ in } L^2((0,1)\times(0,T)) \text{ and } u_{\varepsilon_n} \to u \text{ in } L^2((0,1)\times(0,T)).$$

Convergence to solutions of intermediate-slip equation

Theorem

For fixed positive Re, σ, *and* $\{\beta_n\} \to 0$ *let* $\{(\bar{h}_n, \bar{u}_n)\}$ *be a sequence of global weak solutions. Define*

$$h_n(x,t) := \bar{h}_n\left(x, \frac{t}{\beta}\right),\ u_n(x,t) := \frac{1}{\beta}\,\bar{u}_n\left(x, \frac{t}{\beta}\right),\ (x,t) \in (0,1)\times(0,\infty).$$

Then $\exists\, h > 0$ *and a subsequence of* (h_n, u_n) *such that, for any* $T > 0$,

$$\begin{aligned} h_n &\to h \quad \textit{in} \quad L^2(0,T; H^1(0,1)) \cap C([0,1]\times[0,T]), \\ u_n &\rightharpoonup u := h\partial_x(\sigma\partial_{xx}h - \Pi(h)) \quad \textit{in} \quad L^2((0,1)\times(0,T)), \end{aligned}$$

and h *is a unique smooth solution to the intermediate-slip equation.*

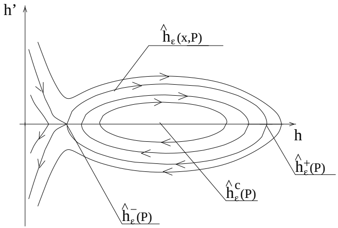
Stationary Solutions (on a way to coarsening)

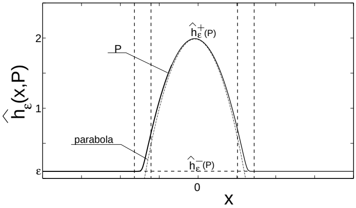
Theorem (Bertozzi et al. '00)

Any lubrication model considered on $\mathbb{R}$ posses a family of stationary solutions with positive nonconstant height profile $\hat{h}_\varepsilon(x, P)$ parameterized by $P \in (0, P_{max}(\varepsilon))$:

$$\partial_{xx}\hat{h}_\varepsilon(x,\ P) = \Pi_\varepsilon(\hat{h}_\varepsilon(x,\ P)) - P.$$

For the strong-slip model such stationary solutions have identically zero velocity.



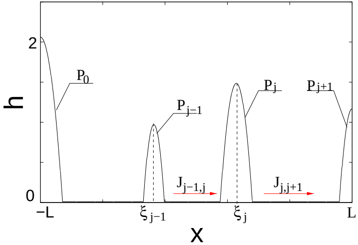


Reduced ODE Models

Weak-slip regime (Glasner and Witelski '03)

$$
\begin{aligned}
\frac{dP_j}{dt} &= C_{P,j}\cdot(J_{j,\,j+1} - J_{j-1,\,j}),\\
\frac{d\xi_j}{dt} &= C_{\xi,j}\cdot(J_{j,\,j+1} + J_{j-1,\,j}),\\
J_{j,j+1} &= (P_{j+1} - P_j)/d_j \quad \text{for} \quad j = 0, ..., N.
\end{aligned}
$$

- P_j, ξ_j–pressure and position of j-th droplet
- $J_{j,\,j+1}$–flux between j-th and $j+1$-th droplets
- $d_j = \xi_{j+1} - \xi_j$.



Geometric Reduction Approach via Mielke&Zelik '08

- No-slip lubrication equation

$$\begin{aligned} \partial_t h + \mathbb{F}_\varepsilon(h) &= 0 \quad \text{with} \quad \mathbb{F}_\varepsilon(h) := \partial_x\Big(h^3 \partial_x\left(\partial_{xx} h - \Pi_\varepsilon(h)\right)\Big), \\ \partial_{xxx} h &= 0 \quad \text{and} \quad \partial_x h = 0 \quad \text{at} \quad x = \pm L. \end{aligned}$$

- Define set of pressures and positions $\mathbb{B}_\varepsilon \subset \mathbb{R}^{2N}$ as

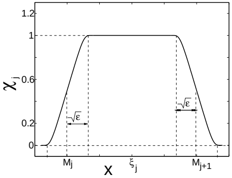
$$\mathbb{B}_\varepsilon = \Big\{ \mathbf{s} = (P_0,\ P_1, ...,\ P_N, \xi_1, \xi_2 ..., \xi_{N-1}) \in \mathbb{R}^{2N} : P_j \in (P_*,\ P^*);$$
$$-L < \xi_1 < ... < \xi_{N-1} < L;\ \xi_i - \xi_{i-1} - 4\,d > 2\sqrt{\varepsilon},\ i = 1, ..., N \Big\},$$

where $\xi_0 := -L$, $\xi_N := L$.

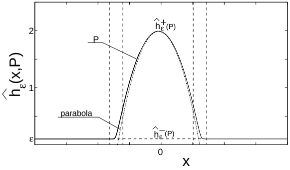
'Approximate Invariant' Manifold I

Define a mapping $\mathbf{m}_\varepsilon : \mathbb{B}_\varepsilon \to L^\infty(-L,\, L)$:

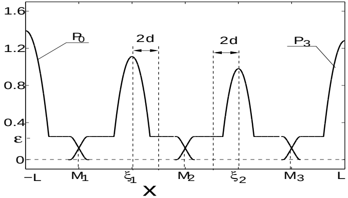
$$\forall \mathbf{s} \in \mathbb{B}_\varepsilon \;\; \mathbf{m}_\varepsilon(\mathbf{s})(x) := \sum_{j=0}^{N} \chi_j(\mathbf{s})(x)\hat{h}_\varepsilon(x - \xi_j,\, P_j)$$



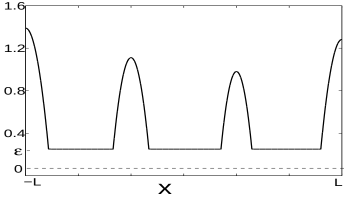
Function $\chi_j(\mathbf{s})(x)$



Steady state $\hat{h}_\varepsilon(x,\, P)$



Truncated pulses $\chi_j(\mathbf{s})(x)\hat{h}_\varepsilon(x - \xi_j,\, P_j)$, $j = 0...3$



Their sum $\mathbf{m}_\varepsilon(\mathbf{s})(x)$

'Approximate Invariant' Manifold II

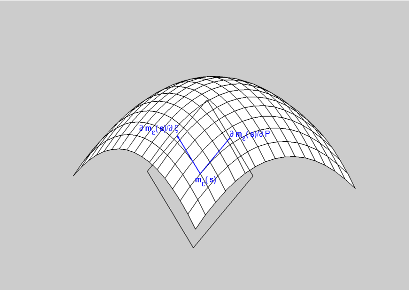
- Image of $\mathbf{m}_\varepsilon$ is $2N$-dimensional submanifold $\mathbb{P}_\varepsilon$ in L^∞
- Mapping $\mathbf{m}_\varepsilon$ is a diffeomorphism between $\mathbb{B}_\varepsilon$ and $\mathbb{P}_\varepsilon$
- Tangent space $\mathbb{T}_{\mathbf{m}}\mathbb{P}_\varepsilon = \mathrm{span}\{\phi_0(\mathbf{s}), \phi_1(\mathbf{s}), ..., \phi_{2N-1}(\mathbf{s})\}$, where

$$\phi_j(\mathbf{s}) := \frac{\partial \mathbf{m}_\varepsilon(\mathbf{s})}{\partial P_j} \quad \text{for} \quad j = 0, ..., N;$$

$$\phi_{N+j}(\mathbf{s}) := \frac{\partial \mathbf{m}_\varepsilon(\mathbf{s})}{\partial \xi_j} \quad \text{for} \quad j = 1, ..., N-1.$$

- For every $\mathbf{m} \in \mathbb{P}_\varepsilon$ and sufficiently small $\varepsilon > 0$ one has

$$\left|\left|\mathbb{F}_\varepsilon(\mathbf{m})\right|\right|_{L^\infty(-L,\,L)} \le \mathrm{const}\,\varepsilon^{3/2}.$$



Projection on the Manifold

Proposition

For every $\mathbf{s} \in \mathbb{B}_\varepsilon$ *there exist 'adjoint' functions*

$$\bar{\psi}_0(\mathbf{s}),\ \bar{\psi}_1(\mathbf{s}), \ldots, \bar{\psi}_{2N-1}(\mathbf{s}) \in C^\infty(-L,\ L),$$

such that for all sufficiently small $\varepsilon > 0$ *and every* $j,\ k \in \{0, \ldots 2N-1\}$ *one has*

$$\big(\bar{\psi}_j(\mathbf{s}),\ \phi_k(\mathbf{s})\big) = \delta_{j,\,k}.$$

For every $\mathbf{m} \in \mathbb{P}_\varepsilon$ define a linear operator $P_\mathbf{m}$ acting on $v \in L^\infty(-L,\ L)$ as

$$\boxed{P_\mathbf{m}\, v := \sum_{j=0}^{2N-1} \big(\bar{\psi}_j(\mathbf{s}),\ v\big)\, \phi_j(\mathbf{s}).}$$

Decomposition Near the Manifold

Theorem

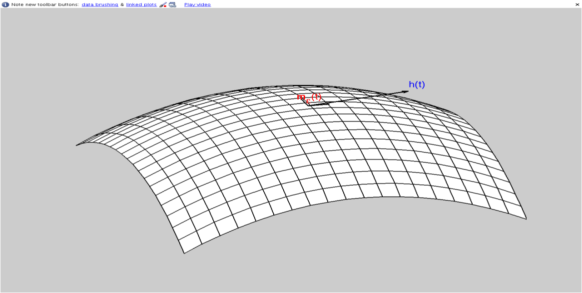
$\exists$ nonlinear differentiable function $\pi_\varepsilon : \mathcal{O}_{\delta_\varepsilon}(\mathbb{P}_\varepsilon) \setminus \mathcal{O}_{\delta_{1,\varepsilon}}(\partial\mathbb{P}_\varepsilon) \to \mathbb{P}_\varepsilon$ such that

$$P_{\mathbf{m}}\, v \equiv 0 \quad \textit{for all} \quad h \in \mathcal{O}_{\delta_\varepsilon}(\mathbb{P}_\varepsilon) \setminus \mathcal{O}_{\delta_{1,\varepsilon}}(\partial\mathbb{P}_\varepsilon),$$

where we denote $\mathbf{m} := \pi_\varepsilon(h)$ and $v := h - \pi_\varepsilon(h)$.

- Applying $P_{\mathbf{m}(t)}$ and $I - P_{\mathbf{m}(t)}$ to the lubrication equation gives

$$\partial_t h + \mathbb{F}_\varepsilon(h) = 0 \iff \begin{cases} \partial_t v + \mathbb{F}_\varepsilon{}'(\mathbf{m}(t))v(t) = h(\mathbf{m}(t),\ v(t),\ \mathbf{m}'(t)) \\ \mathbf{m}'(t) = f(\mathbf{m}, v) \end{cases}$$



Equation on the Manifold

- Put formally $v(t) \equiv 0$ for $t > 0$ then

$$\mathbf{m}'(t) = f(\mathbf{m}(t), 0) \iff \sum_{i=0}^{i=N} \phi_i(\mathbf{s}) \frac{dP_i}{dt} + \sum_{i=N+1}^{i=2N-1} \phi_i(\mathbf{s}) \frac{d\xi_i}{dt} = -P_{\mathbf{m}} \mathbb{F}_\varepsilon(\mathbf{m}).$$

- Taking the standard scalar product in $L^2(-L, L)$ with $\bar{\psi}_j(\mathbf{s})$ gives

$$\begin{aligned} \frac{dP_j}{dt} &= C_{P,j} \cdot (J_{j,\,j+1} - J_{j-1,\,j}), \\ \frac{d\xi_j}{dt} &= -C_{\xi,j} \cdot (J_{j,\,j+1} + J_{j-1,\,j}),\ j = 0, ..., N, \end{aligned}$$

where

$$C_{P,j} := -1 \Big/ \int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\partial \hat{h}_\varepsilon(x - \xi_j,\, P_j)}{\partial P}\, dx,$$

$$C_{\xi,j} := \frac{\int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\hat{h}_\varepsilon(x-\xi_j, P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(x-\xi_j, P_j)^3}\, dx}{2\int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\left(\hat{h}_\varepsilon(x-\xi_j, P_j) - \hat{h}_\varepsilon^-(P_j)\right)^2}{\hat{h}_\varepsilon(x-\xi_j, P_j)^3}\, dx}.$$

Equation on the Manifold

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where

$$\begin{aligned} J_{j-1,\, j} &:= J(\mathbf{s})(\theta_j), \;\; j = 1, ..., N-1, \\ J_{-1,\, 0} &:= -J_{0,\, 1}, \;\; J_{N,\, N+1} := -J_{N-1,\, N} \end{aligned}$$

with θ_j being some point in $(M_j - \sqrt{\varepsilon},\, M_j + \sqrt{\varepsilon})$ and

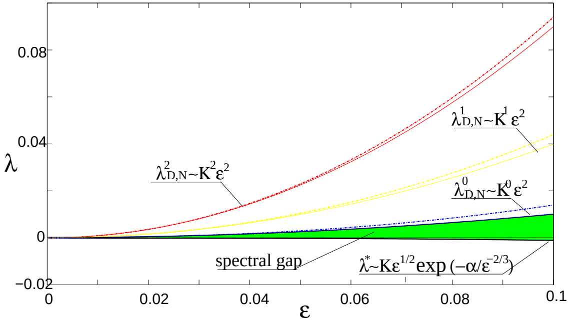
$$J(\mathbf{s}) := (\mathbf{m}_\varepsilon(\mathbf{s}))^3 \partial_x \Big(-\Pi_\varepsilon(\mathbf{m}_\varepsilon(\mathbf{s})) + \partial_{xx} \mathbf{m}_\varepsilon(\mathbf{s}) \Big).$$

Spectral Asymptotics

Mielke&Zelik: Show existence of an invariant attracting manifold in $\mathcal{O}_{\delta_\varepsilon}(\mathbb{P}_\varepsilon)$ diffeomorphic to the 'approximate invariant' one $\mathbb{P}_\varepsilon$

We show the spectral assumption:

- The spectrum of the linearized evolution equation on a droplet steady state has an exponentially small eigenvalue.
- Between it and the rest of the spectrum there is a gap $\left(0,\ \left[\frac{\pi}{4(L-A/P)}\varepsilon\right]^2\right)$ for all sufficiently small $\varepsilon > 0$.



Linear Eigenvalue Problem

- Droplet stationary solution $h_{0,\varepsilon}(x)$
- Hilbert space $W_\varepsilon := H^2(-L/\varepsilon,\, L/\varepsilon) \cap H_0^1(-L/\varepsilon,\, L/\varepsilon)$
- Coefficient functions $r_\varepsilon(x) := -\Pi'(h_{0,\varepsilon}(x)), \quad f_\varepsilon(x) := (h_{0,\varepsilon}(x))^{-3}$

Theorem (Laugesen&Pugh '00)

Consider a symmetric eigenvalue problem

$$h \in W_\varepsilon,\, \lambda \in \mathbb{R} \, : \quad \int_{-L/\varepsilon}^{L/\varepsilon} (h''w'' - r_\varepsilon h' w' - \lambda f_\varepsilon h\, w)\, dx = 0,\, \forall w \in W_\varepsilon.$$

For a fixed $\varepsilon > 0$ *there exist sequences* $\{\lambda_\varepsilon^*,\, \lambda_\varepsilon^0,\, \lambda_\varepsilon^1, \ldots\}$, $\{h_\varepsilon^*,\, h_\varepsilon^0,\, h_\varepsilon^1, \ldots\}$*:*

(i) *for each* $j \in \{*,\, 0,\, 1, \ldots\}$ *the pair* $[h_\varepsilon^j,\, \lambda_\varepsilon^j]$ *is a solution;*

(ii) $\lambda_\varepsilon^* \le \lambda_\varepsilon^0 \le \lambda_\varepsilon^1 \le \lambda_\varepsilon^2 \le \ldots \to \infty$;

(iii) *set* $\{h_\varepsilon^j\}$, $j \in \{*,\, 0,\, 1, \ldots\}$ *forms an orthonormal basis in* $L^2(-L/\varepsilon,\, L/\varepsilon)$*;*

Spectral Gap (K., Recke and Wagner '11)

Theorem (algebraic eigenvalues)

For every $j \in \mathbb{N}_0$ *there exist positive numbers* ε^j, δ^j *and functions* $\lambda_N^j,\ \lambda_D^j \in C^1((0,\ \varepsilon^j), \mathbb{R})$ *such that for all* $\varepsilon \in (0,\ \varepsilon^j)$*:*

(i) $\lambda_N^j(\varepsilon) \in \sigma_\varepsilon$ *and* $\lambda_D^j(\varepsilon) \in \sigma_\varepsilon$,

(ii) $\left|\lambda_N^j(\varepsilon) - \left(\frac{\pi(2j+1)}{2(L-A/P)}\varepsilon\right)^2\right| = o_1(\varepsilon^2), \quad \left|\lambda_D^j(\varepsilon) - \left(\frac{\pi(2j+1)}{2(L-A/P)}\varepsilon\right)^2\right| = o_2(\varepsilon^2),$

(iii) *If* $\lambda \in \sigma_\varepsilon$ *and* $\left|\lambda - \left(\frac{\pi(2j+1)}{2(L-A/P)}\varepsilon\right)^2\right| \le \delta^j\varepsilon^2$ *then* $\lambda = \lambda_N^j(\varepsilon)$ *or* $\lambda = \lambda_D^j(\varepsilon)$.

Theorem (exponentially small eigenvalue)

There exist positive constants $c^*,\ \alpha,\ \varepsilon^*, \delta^*$ *and function* $\lambda^* \in C^1((0,\ \varepsilon^*), \mathbb{R})$ *such that for all* $\varepsilon \in (0,\ \varepsilon^*)$*:*

(i) $\lambda^*(\varepsilon) \in \sigma_\varepsilon$

(ii) $|\lambda^*(\varepsilon)| \le c^*\varepsilon^{1/2}\exp\left(-\frac{\alpha}{\varepsilon^{2/3}}\right)$,

(iii) *If* $\lambda \in \sigma_\varepsilon$ *and* $|\lambda| \le \delta^*\varepsilon^2$ *then* $\lambda = \lambda^*(\varepsilon)$.

Reduced ODEs for the Strong-slip Model (K. '14)

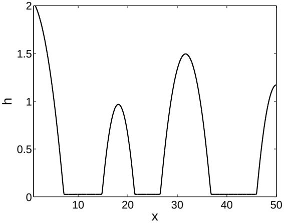
$$
\begin{aligned}
\dot{\xi}_j &= C_{\xi_j}(J_{j,j+1} + J_{j,j-1}),\\
\dot{P}_j &= C_{P_j}(J_{j,j+1} - J_{j,j-1}),\\
J_{j,j+1} &= \frac{[P_{j+1} - P_j] - \nu I(\dot{\xi}_{j+1} + \dot{\xi}_j)}{d_j + 2I\nu\beta}, \quad j = 0, ..., N;
\end{aligned}
$$

where

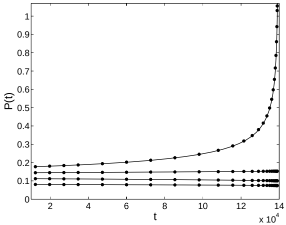
$$
C_{\xi_j} = \frac{-I\beta}{2A/(P_j\sqrt{\sigma}\beta\nu) + 2I}, \quad C_{P_j} = \varepsilon\frac{4A^3}{3P_j^3}.
$$

Numerical comparison

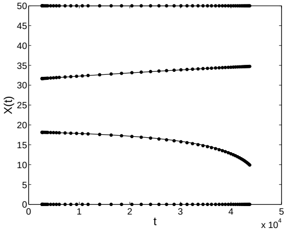
initial profile



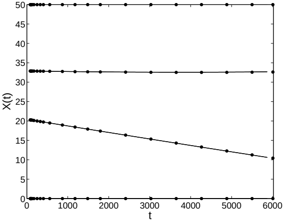
intermidiate-slip



$\beta = 5$



$\beta = \infty$



Exactly solvable collision/absorption model I

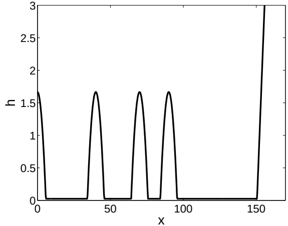
- Consider leading order migration system for free films ($\beta = \infty$):

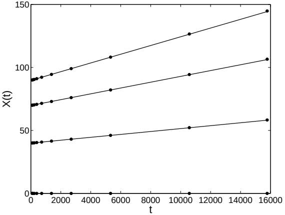
$$\dot{X}_0 = \dot{X}_N = \dot{P}_j = 0 \qquad \text{for } j = 0, ..., N;$$

$$\dot{X}_{j+1} - 2\dot{X}_j + \dot{X}_{j-1} = \frac{P_{j+1} - P_{j-1}}{\nu I} \qquad \text{for } j = 2, ..., N-1;$$

$$P_N(0) = \bar{p} \text{ with } 1 \gg p \gg \bar{p} \text{ and } P_j(0) = p \qquad \text{for } j = 0, ..., N-1.$$

- The model reproduces absorption by a huge droplet via subsequent collisions an array of N small droplets.



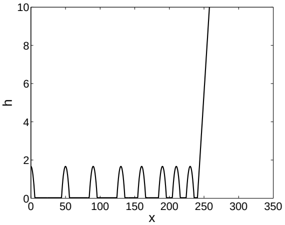


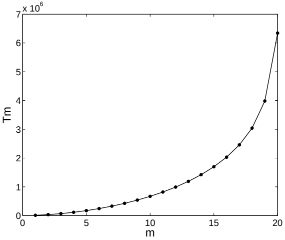
Exactly solvable collision/absorption model II

- The exact solution between subsequent collisions is given by

$$d_j(T_c) = d_j(0) + \frac{d_n}{N-1}, \quad j = 1, ..., N-1$$
$$\text{where } T_c = \frac{d_N N}{(N-1)B} \qquad \text{with } B = \frac{p - \bar{p}}{\nu I}.$$

- **Remark:** Compare with heuristic breath figure models in physics! (Derrida et. al '91, Bray et. al '94)





Coarsening rates law: continuum case

- Denote by $n(d)$ a relative number of droplets with the distances larger or equal d, i.e.

$$n(d) = 1 - \int_0^d f(x)\,dx$$

- Then the exact coarsening rate law is given by:

$$T(d) = \frac{1}{B}\int_0^d n(x)\ln\left[\frac{n(x)}{n(d)}\right]\,dx.$$

- The discrete coarsening law can be recovered by substitution

$$f(d) = \sum_{m=1}^{k} \frac{i_m}{N}\delta(d - d_m).$$

Examples of coarsening rates

- Consider a family of distributions

$$f(x) = \frac{A^{\alpha}}{x^{1+\alpha}} \quad \text{with } A,\, \alpha > 0.$$

- The exact coarsening law for $\alpha = 1$:

$$n(t) = \exp\left[1 - \sqrt{1 + 2Bt/A}\right]$$

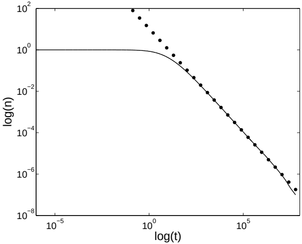
- For $\alpha \neq 1$ the asymptotic coarsening law as $t \to \infty$:

$$n(t) \sim \begin{cases} \left(\dfrac{tB(\alpha-1)^2}{\alpha A}\right)^{\frac{\alpha}{\alpha-1}}, & \text{if } \alpha < 1 \\ \exp\left\{-\dfrac{tB(\alpha-1)}{\alpha A}\right\}, & \text{if } \alpha > 1 \end{cases} .$$

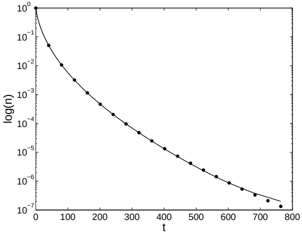
- <u>**Conclusion:**</u> For $0 < \alpha < 1$ the coarsening rates are algebraic while they become exponential for $\alpha \in [1, +\infty)$.

Numerical comparison

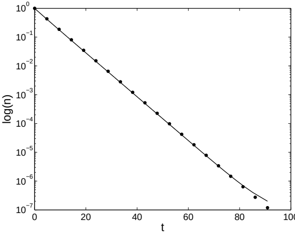
$\alpha = 1/2$



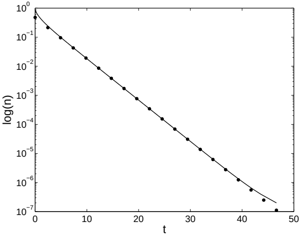
$\alpha = 1$



$\alpha = 20$



Gaussian



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Thank you for your attention!