Asymptotic stability of kinks for nonlinear relativistic Ginzburg-Landau equation

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Introduction

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The first results in this direction were obtained by numerical simulation in 1965 by Zabusky and Kruskal for the Korteweg-de Vries (KdV) equation. In 1967 Gardner, Greene, Kruskal, and Miura used the inverse scattering transform to solve the KdV equation analytically. These results were extended to other integrable equations by Its, Khruslov, Shabat, Zakharov, and other.

The asymptotic stability of solitons for a nonlinear Schrödinger equation (NLS) with small initial data and small coefficient of the nonlinear term was proved by Soffer and Weinstein (1990), (1992). Later, Buslaev and Perelman (1995) and Buslaev and Sulem (2003) established this result in the more difficult instance. Now there are a lot of works concerning non-relativistic nonlinear equations.

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[1] A.Komech, EK, Scattering of solitons for NLS coupled to a particle, *Russian J.Math.Phys.* (2006)

[2] V.Buslaev, A.Komech, EK, D.Stuart, On asymptotic stability of solitary waves in NLS, *Comm.Partial Diff.Eqns.* (2008)

[3] EK, On asymptotic stability of solitary waves in DNLS, *Nonlin.Analysis Series A: Theory, Methods and Application* (2009)

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[5] A.Komech, EK, H.Spohn, Scattering of solitons for Dirac equation coupled to a particle, *J. Math. Anal. Appl.* (2011).

[6] A.Komech, EK, D.Stuart, On asymptotic stability of solitons in a NLS *Comm.Pure and Applied Analysis* (2012).
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Model: nonlinear relativistic wave equation

$$\ddot{\psi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R},$$
$$F(\psi) = -U'(\psi)$$

Condition U1

$$U(\psi) > 0, \quad \psi \neq \pm 1;$$
$$U(\psi) = \frac{m^2}{2}(\psi \mp 1)^2 + \mathcal{O}(|\psi \mp 1|^K), \quad \psi \to \pm 1.$$
$$K \geq 8$$

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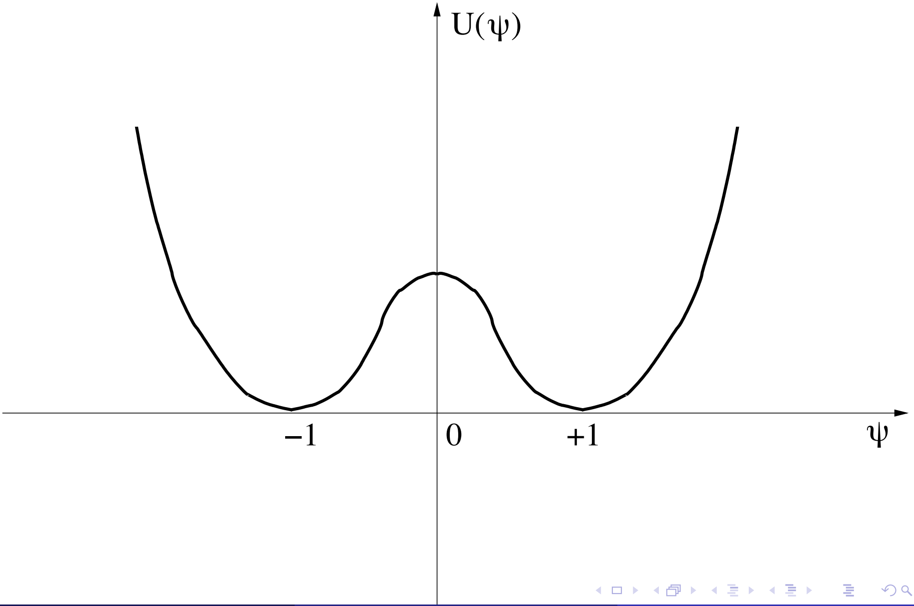
The classic Ginzburg-Landau potential: $U(\psi) = \frac{(\psi^2-1)^2}{4}$, $F(\psi) = -\psi^3 + \psi$ does not satisfy **U1** since $K = 3$ then.

Potential of Ginzburg-Landau type



Stationary equation

$$s'' - U'(s) = 0$$

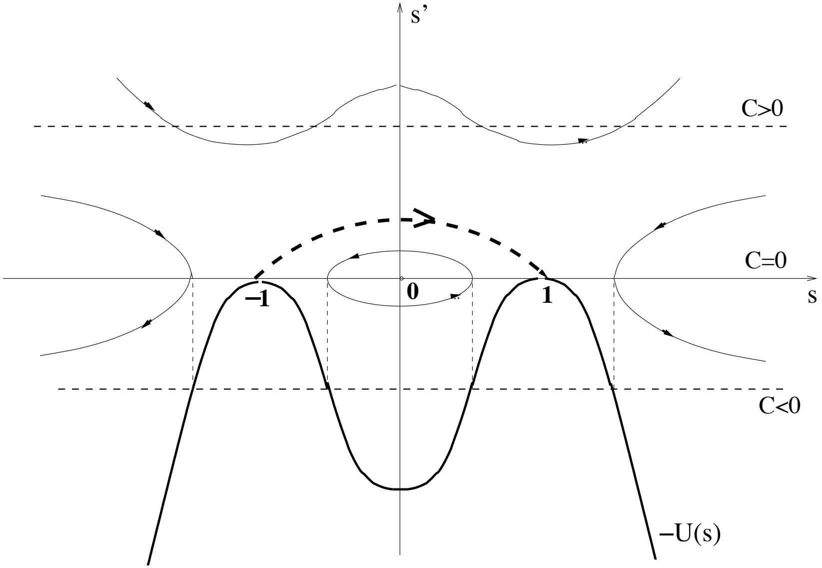
Constant solutions of the stationary equation are

$$s \equiv \pm 1, \quad s \equiv 0$$

Integrating stationary equation we get

$$\frac{(s')^2}{2} - U(s) = C$$

Phase portrait of stationary equation $\frac{(s')^2}{2} - U(s) = C$



Kink: nonconstant finite energy solution

$$s(x) \to \pm 1 \quad \text{as} \quad x \to \pm\infty$$

Conditions **U1** implies

$$(s(x) \mp 1)'' \sim m^2(s(x) \mp 1), \quad x \to \pm\infty$$

$$s(x) \mp 1 \sim Ce^{-m|x|}, \quad x \to \pm\infty$$

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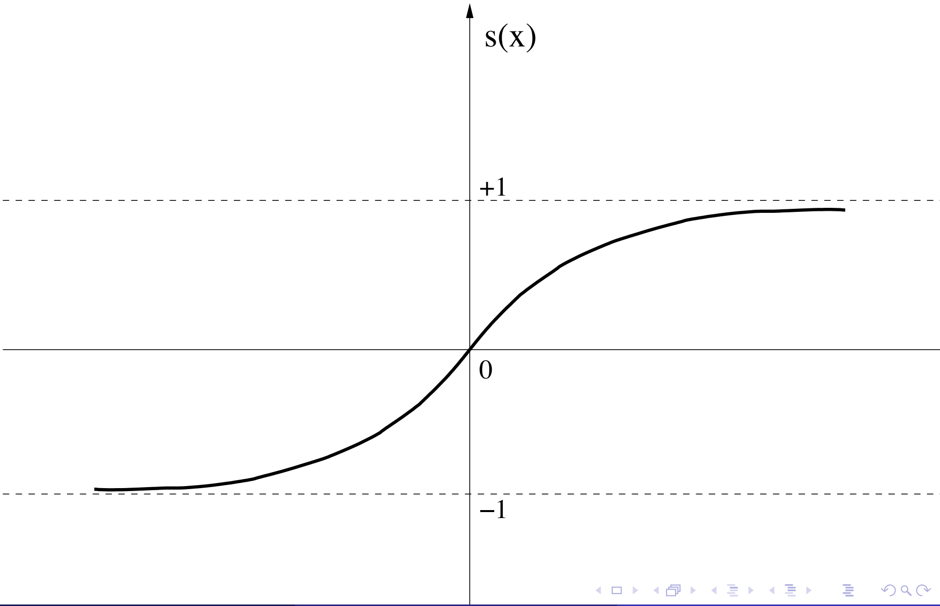
In the case of Ginzburg-Landau potential $\quad s(x) = \tanh \frac{x}{\sqrt{2}}$

Plot of kink



Hamilton equation

In the vector form, equation (1) reads

$$\begin{cases} \dot{\psi}(x,t) = \pi(x,t) \\ \\ \dot{\pi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R} \end{cases}$$

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$$\dot Y(t) = \mathcal{F}(Y(t)), \quad Y(t) = \begin{pmatrix} \psi(t) \\ \pi(t) \end{pmatrix} \qquad (2)$$

It is a Hamilton equation, i.e.

$$\dot Y = J\mathcal{D}\mathcal{H}(Y), \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\text{where} \ \ \mathcal{H}(Y) = \int \Big[\frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + U(\psi(x))\Big]\, dx$$

and $\mathcal{D}\mathcal{H}$ is the Fréchet derivative of the Hamilton functional $\mathcal{H}$.

Traveling waves or soliton solutions

$$S_{q,v}(t) = (\psi_v(x - vt - q), \pi_v(x - vt - q))$$

$$q, v \in \mathbb{R}, \qquad |v| < 1$$

$$\psi_v(x) = s(\gamma x), \quad \pi_v = -v\psi_v'(x)$$

$$\gamma = 1/\sqrt{1 - v^2}$$

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Definition: A solitary manifold is the set

$$\mathcal{S} := \{S_{q,v} : \; q, v \in \mathbb{R}, \; |v| < 1\}$$

Linearization at soliton

$$Y(t) = S_{q,v}(t) + X(t)$$

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$$V_v(x) \sim C(s(\gamma x) \mp 1)^{K-2} \sim Ce^{-(K-2)m\gamma|x|}, \quad x \to \pm\infty.$$

Spectral properties of linearized equation

The determinant of A_v is the Schrödinger operator

$$H_v = -(1 - v^2)\frac{d^2}{dx^2} + m^2 + V_v$$

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$$\varphi_0(x) = s'(\gamma x)$$

Condition U2: The edge point m^2 is not resonance of H_v

Point m^2 is a resonance if there exist a nonzero solution $\psi \in L^\infty(\mathbb{R})$ to

$$H_v\psi = m^2\psi,$$

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where $\frac{m^2}{4} < \mu < m^2$

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Condition U4: The Fermi Golden Rule holds

$$\int \varphi_{4\mu}(x) F''(s(x)) \varphi_\mu^2(x) dx \neq 0$$

Theorem (Main result)

Let conditions **U1** $-$ **U4** *hold, and* $Y_0 = S_{q_0,v_0} + X_0$*, where* X_0 *is sufficiently small. Then*

$$Y(x,t) = (\psi_{v_\pm}(x - v_\pm t - q_\pm), \pi_{v_\pm}(x - v_\pm t - q_\pm))$$

$$+W_0(t)\Phi_\pm + r_\pm(x,t), \quad t \to \pm\infty.$$

Here $\Phi_\pm \in E = H^1 \oplus L^2$*, and*

$$\|r_\pm(t)\|_E = \mathcal{O}(|t|^{-\nu}), \quad t \to \pm\infty, \quad \nu > 0$$

Classic Ginzburg-Landau potential

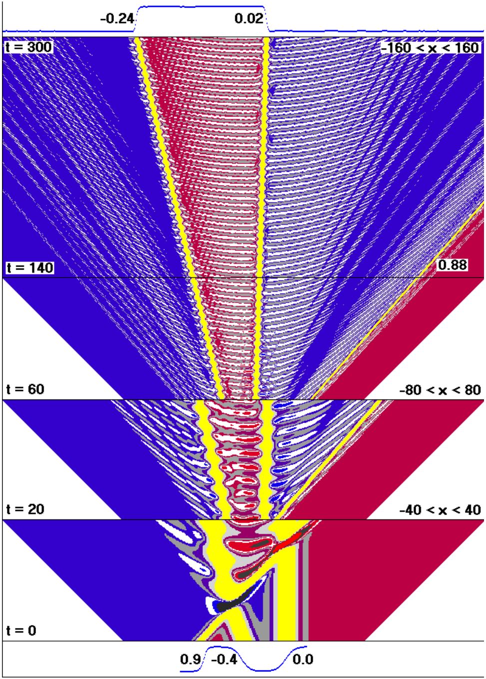
$$V_v(x) = \frac{-3}{\cosh^2(\gamma x/\sqrt{2})}$$

- The continuous spectrum is $[2, \infty)$.
- The discrete spectrum is $\{0; 3/2\}$.
- The corresponding Fermi Golden Rule holds

There exists resonance: the function

$$\psi(x) = 1 - 3\tanh^2(\gamma x/\sqrt{2}) \in L^\infty(\mathbb{R})$$

is the solution to $H_v\psi = 2\psi$. Then condition **U2** fails.


-0.24
0.02
t = 300
-160 < x < 160
t = 140
0.88
t = 60
-80 < x < 80
t = 20
-40 < x < 40
t = 0
0.9
-0.4
0.0

Methods

For the proof we develop the approach of [1]-[2] for Schrödinger equation

- *Symplectic projection* of the trajectory onto the solitary manifold $\mathcal{S}$
- *Modulation equations* for the parameters of the symplectic projection (Dynamics along $\mathcal{S}$)
- Linearization of the transversal dynamics
- Dispersion decay of linearized dynamics
- Poincaré normal form
- Bounds for majorants.

[1] V.S.Buslaev, G.S.Perelman, On the stability of solitary waves for nonlinear Schrödinger equations, *Amer.Math.Soc.Trans.* (1995).
[2] V.S.Buslaev, C.Sulem, On asymptotic stability of solitary waves for nonlinear Schrödinger equations, *Ann.Inst.Henri Poincaré, Anal.Non Linéaire* (2003).

Symplectic structure

Consider the tangent space $\mathcal{T}_{S_{q,v}}\mathcal{S}$ of the manifold $\mathcal{S}$ at a point $S_{q,v}$. The vectors

$$\begin{aligned}\tau_1 = \tau_1(v) &:= \partial_v S_{q,v} = (-\psi_v', -\pi_v') \\ \tau_2 = \tau_2(v) &:= \partial_q S_{q,v} = (\partial_v \psi_v, \partial_v \pi_v)\end{aligned}$$

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form a basis in $\mathcal{T}_{S_{q,v}}\mathcal{S}$. The symplectic form Ω

$$\Omega(\tau_1, \tau_2) := \langle \tau_1, J\tau_2 \rangle$$

is nondegenerate on the tangent space $\mathcal{T}_{S_{q,v}}\mathcal{S}$: $\quad \Omega(\tau_1, \tau_2) \neq 0$
i.e. $\mathcal{T}_{S_{q,v}}\mathcal{S}$ is a symplectic subspace.
In a small neighborhood of the soliton manifold $\mathcal{S}$ a "symplectic orthogonal projection" onto $\mathcal{S}$ is well-defined.

Zero eigenspace

$$A_v\tau_1 = 0, \quad A_v\tau_2 = \tau_1$$

Then $Z(t) = c_1\tau_1 + c_2(\tau_1 t + \tau_2)$ is a growing solution to linearized equation

$$\dot{Z}(t) = A_v Z(t)$$

It means the unstable character of dynamics along the solitary manifold.

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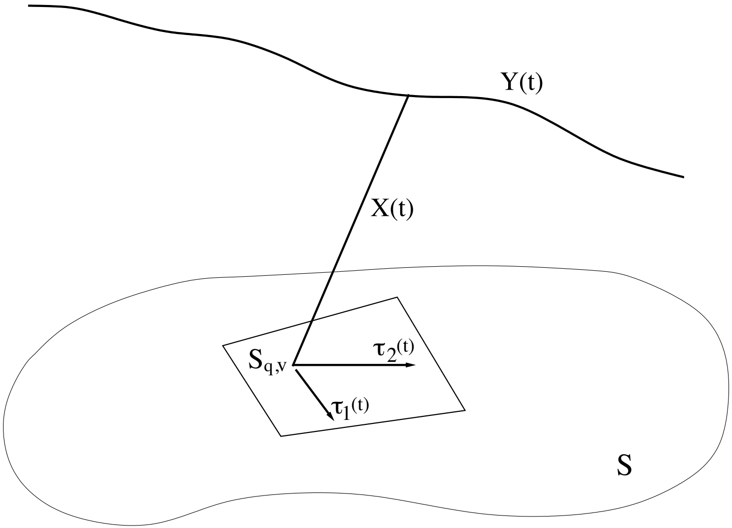
It means the unstable character of dynamics along the solitary manifold. We split a solution to (2) as the sum

$$Y(t) = S_{q(t),v(t)} + X(t),$$

where the symplectic orthogonality condition hold:

$$\Omega(X(t), \tau_1(t)) = \Omega(X(t), \tau_2(t)) = 0$$

Solitary manifold



Transversal dynamics

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Lemma

$$\|X(t)\| \to 0, \quad t \to \pm\infty$$

Main novelties

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2. Estimates of the growth rate of moments of solutions of the nonlinear Klein-Gordon equation
3. The relativistic version of decay estimates of solutions in $L^1 - L^\infty$-norms

Weighted energy decay

Consider Klein-Gordon equation

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   $H = -\frac{d^2}{dx^2} + m^2 - V(x)$

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Definition: Weighted Sobolev spaces

$L^2_\sigma = L^2_\sigma(\mathbb{R}), \quad H^1_\sigma = H^1_\sigma(\mathbb{R}), \quad \sigma \in \mathbb{R}$

$$\|\psi\|_{L^2_\sigma} = \|(1+|x|)^\sigma\psi\|_{L^2} < \infty, \quad \|\psi\|_{H^1_\sigma} = \|(1+|x|)^\sigma\psi\|_{H^1} < \infty,$$

Theorem (Dispersion decay)

Let conditions (1) and (2) hold, and let

$$\Psi(0) = (\psi(0), \dot{\psi}(0)) \in H^1_\sigma \oplus L^2_\sigma, \ \sigma > 5/2$$

Then

$$\|\mathcal{P}_c \Psi(t)\|_{H^1_{-\sigma} \oplus L^2_{-\sigma}} = \mathcal{O}(|t|^{-3/2}), \quad t \to \pm\infty$$

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[2] EK, On long-time decay for modified Klein-Gordon equation, *Comm. Math. Analysis* (2011).

Theorem

There exist potentials $U(\psi)$ satisfying conditions **U1**-**U4**.

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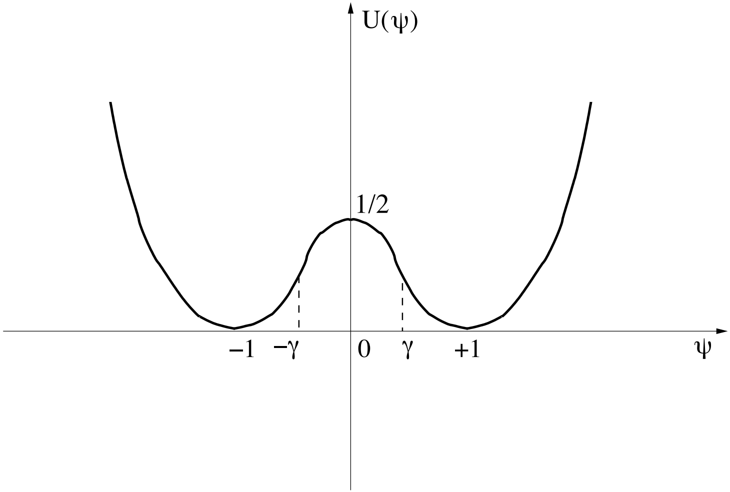
First we consider piece-wise second order polynomials potentials:

$$U_0(\psi) = \begin{cases} \frac{1}{2}(1 - b\psi^2), & |\psi| \le \gamma \\ \frac{d}{2}(\psi \mp 1)^2, & \pm\psi \ge \gamma \end{cases}$$

The condition $U_0(\psi) \in C^1(\mathbb{R})$ implies

$$b = \frac{1}{\gamma}, \quad d = \frac{1}{1-\gamma}, \quad 0 < \gamma < 1.$$

Potential U_0



The kink solution has the form

$$s(x) = \begin{cases} C \sin \sqrt{b} x, & 0 < x \le q, \\ \\ A e^{-\sqrt{d} x} + 1, & x > q, \end{cases}$$

where

$$C = \sqrt{\gamma}, \quad A = (\gamma - 1) \exp\left(\sqrt{\gamma/(1-\gamma)} \arcsin \sqrt{\gamma}\right)$$

$$q = \sqrt{\gamma} \arcsin \sqrt{\gamma}.$$

The kink solution has the form

$$s(x) = \begin{cases} C \sin \sqrt{b} x, & 0 < x \le q, \\ \\ A e^{-\sqrt{d} x} + 1, & x > q, \end{cases}$$

where

$$C = \sqrt{\gamma}, \quad A = (\gamma - 1) \exp(\sqrt{\gamma/(1-\gamma)} \arcsin \sqrt{\gamma})$$

$$q = \sqrt{\gamma} \arcsin \sqrt{\gamma}.$$

The corresponding Schrödinger operator reads

$$H = -\frac{d^2}{dx^2} + W_0(x)$$

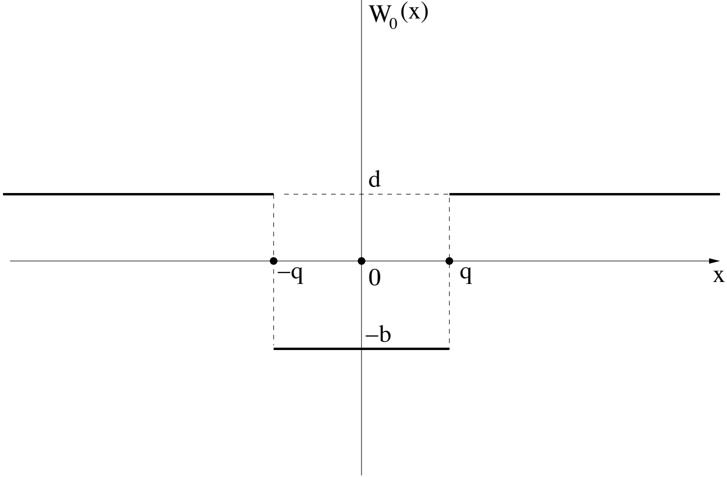


Figure : Potential W_0

$$W_0(x) = U_0''(s(x)) = \begin{cases} -b, & |x| \le q \\ d, & |x| > q \end{cases}$$

The continuous spectrum $\sigma_c = [d, \infty)$. The point 0 is the groundstate since it corresponds to the symmetric positive eigenfunction

$$\varphi_0(x) = s'(x)$$

Therefore, the discrete spectrum $\sigma_d \subset [0, d]$, and the next eigenfunction $\varphi_1(x)$ should be antisymmetric.

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Therefore, the discrete spectrum $\sigma_d \subset [0, d]$, and the next eigenfunction $\varphi_1(x)$ should be antisymmetric. Let γ_k, $k = 1, 2, ...$ be the solution to

$$\frac{\arcsin\sqrt{\gamma_k}}{\sqrt{1-\gamma_k}} = \frac{k\pi}{2}$$

Numerical calculations gives

$$\gamma_1 \sim 0.64643, \quad \gamma_2 \sim 0.8579$$

$$\gamma_3 \sim 0.92472, \quad \gamma_4 \sim 0.95359, \quad \gamma_5 \sim 0.96856...$$

The set has the limit point 1

There is exactly one eigenvalue $\lambda_0 = 0$ if $\gamma \in (0, \gamma_1]$
There are exactly two eigenvalues $\lambda_0 = 0$ and $\lambda_1 \in (0, d)$ if $\gamma \in (\gamma_1, \gamma_2]$
etc.

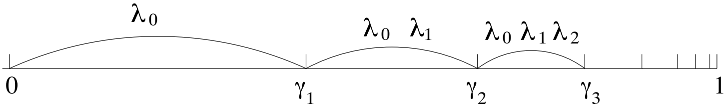

Figure : Spectr

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