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Feynmanons in the Korteweg-de Vries equation

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The Korteweg-de Vries (KdV) equation:

$$\frac{\partial u}{\partial t} - 6 \cdot u \cdot \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad t > 0, \quad -\infty < x < +\infty, \quad u(x,0) = u_0(x)$$

The KdV equation describes a wide range of physical phenomena:

- **nonlinear long waves on “shallow water”;**
- **nonlinear ionic-sound waves in collisionless plasma;**
- **the Fermi-Pasta-Ulam problem.**

Gardner C.S., Greene J. M., Kruskal M. D., Miura R. M. Method for solving the Korteweg – de Vries equation // Phys. Rev. Lett. 1967. V. 19. P. 1095-1097.

What one ought to do in order to solve Cauchy problem for the KdV equation:

1. to determine scattering data for stationary Schrödinger equation:

$$\frac{d^2\psi}{dx^2} + (\lambda - u_0(x)) \cdot \psi = 0$$

$$|a_0(p)|^2 + |b_0(p)|^2 = 1 \qquad c_m = \lim_{x \to +\infty} [\psi_m \cdot \exp(x \cdot \sqrt{-\lambda_m})]$$

2. to solve the Gelfand-Levitan-Marchenko (GLM) equation:

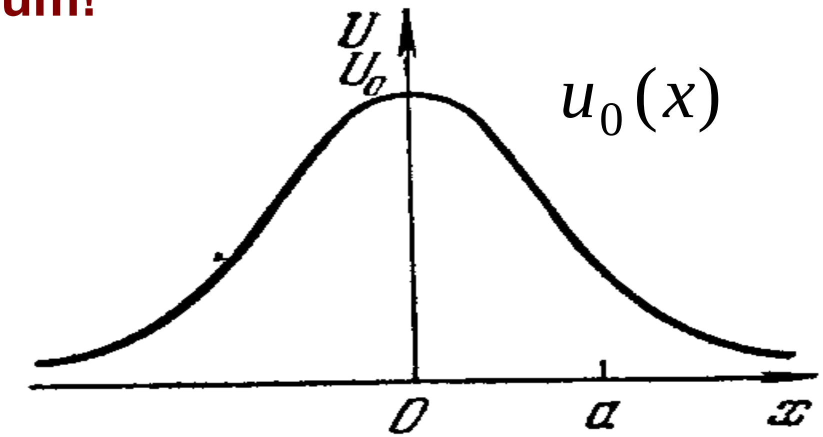
$$K(x, y; t) + B(x + y; t) + \int_x^{+\infty} B(y + z; t) \cdot K(x, z; t) \cdot dz = 0$$

3. To find the exact solution of the KdV equation as follows:

$$u(x, t) = -2 \cdot \frac{\partial K(x, x; t)}{\partial x}$$

Consider the situation when initial condition for the KdV equation is the potential hump:

This potential has no discrete spectrum!



$b_0(p)$ — **reflection coefficient**

In this case the kernel of the GLM equation:

$$B(x,t) = \int_{-\infty}^{+\infty} b_0(p) \cdot \exp(i \cdot p \cdot x + 8 \cdot i \cdot p^3 \cdot t) \cdot \frac{dp}{2 \cdot \pi}$$

The kernel of the GLM equation obeys to the linearized KdV equation:

$$\frac{\partial B}{\partial t} + 8 \cdot \frac{\partial^3 B}{\partial x^3} = 0$$

Appearance of Feynman integral

Rewriting equation for the kernel of the GLM equation as nonstationary equation of Schrödinger-like type:

$$i \cdot \frac{\partial \mid B>}{\partial t} = \hat{H} \mid B>$$

$$\hat{H} = -8 \cdot \hat{p}^{\,3}$$ — Hamiltonian

$$\hat{p} = -i \cdot \partial / \partial x$$ — operator of momentum

Formal solution:

$$\mid B(t) >= \exp(-i \cdot \hat{H} \cdot t) \mid B(0) >$$

$$< x \mid B(0) > \equiv B(x,0) = \int_{-\infty}^{+\infty} b_0(p) \cdot \exp(i \cdot p \cdot x) \cdot \frac{dp}{2 \cdot \pi}$$

On the other side using Green's function we find:

$$< x \mid B(t) > \equiv B(x,t) = \int_{-\infty}^{+\infty} G(x - x'; t) \cdot B(x',0) \cdot dx'$$

$$G(x-x';t)=<x|\exp(-i\cdot\hat{H}\cdot t)|x'>$$

Partition of time segment [0,t] on N+1 equal segments:

$$[0,t]=\bigcup_{j=0}^{N}[t_j,t_{j+1}]$$

$$\Delta t_j=t_{j+1}-t_j=\Delta t \quad t_0=0 \quad t_{N+1}=t$$

$$\Delta t=\frac{t}{N+1}$$

Feynman formula:

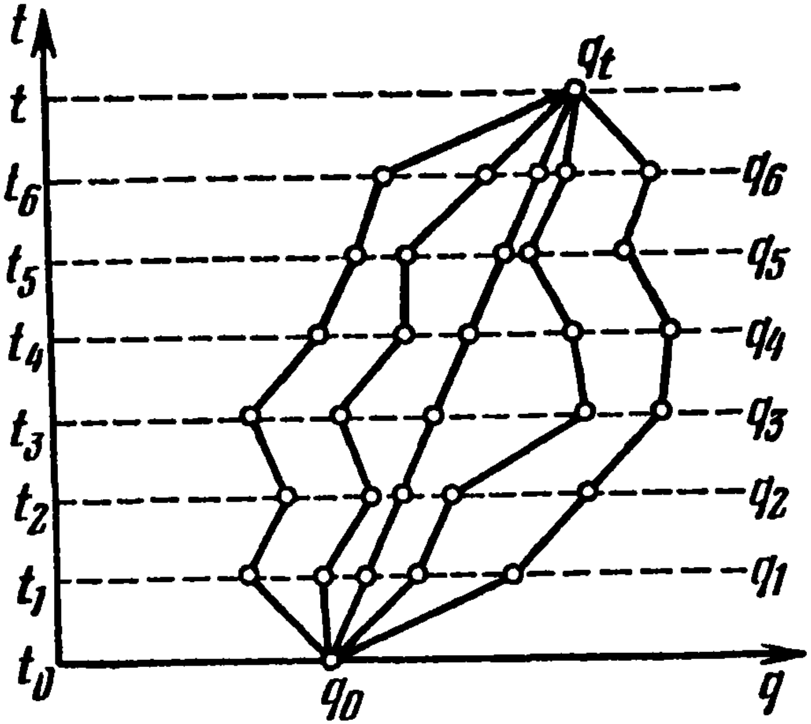
$$\exp(-i\cdot\hat{H}\cdot t)=\exp(-i\cdot\hat{H}\cdot\Delta t_N)\cdot\exp(-i\cdot\hat{H}\cdot\Delta t_{N-1})\cdot\ldots\cdot\exp(-i\cdot\hat{H}\cdot\Delta t_0)$$

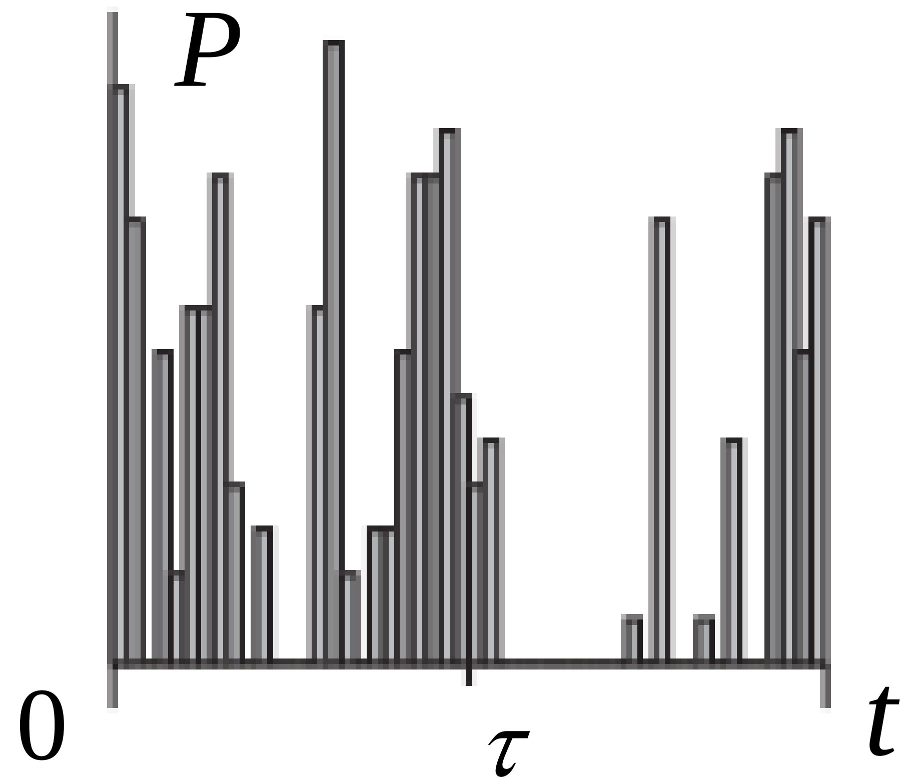
$$<x|\exp(-i\cdot\hat{H}\cdot t)|x'>=\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty}<x|\exp(-i\cdot\hat{H}\cdot\Delta t_N)|Q_N>\cdot\ldots\cdot<Q_1|\exp(-i\cdot\hat{H}\cdot\Delta t_0)|x'>dQ_1\ldots dQ_N$$

$$<Q_{j+1}|\exp(8\cdot i\cdot\Delta t_j\cdot\hat{p}^3)|Q_j>=\int_{-\infty}^{+\infty}\exp[i\cdot P_{j+1}\cdot(Q_{j+1}-Q_j)+8\cdot i\cdot\Delta t_j\cdot P_{j+1}^3]\cdot\frac{dP_{j+1}}{2\cdot\pi}$$

$$<x|\exp(-i\cdot\hat{H}\cdot t)|x'>=\int_{R^{2N+1}}\exp\left(i\cdot\sum_{j=0}^{N}[P_{j+1}\cdot(Q_{j+1}-Q_j)+8\cdot\Delta t\cdot P_{j+1}^3]\right)\cdot\frac{d^{N+1}P\cdot d^NQ}{(2\cdot\pi)^{N+1}}$$

Feynman integral or path integral:

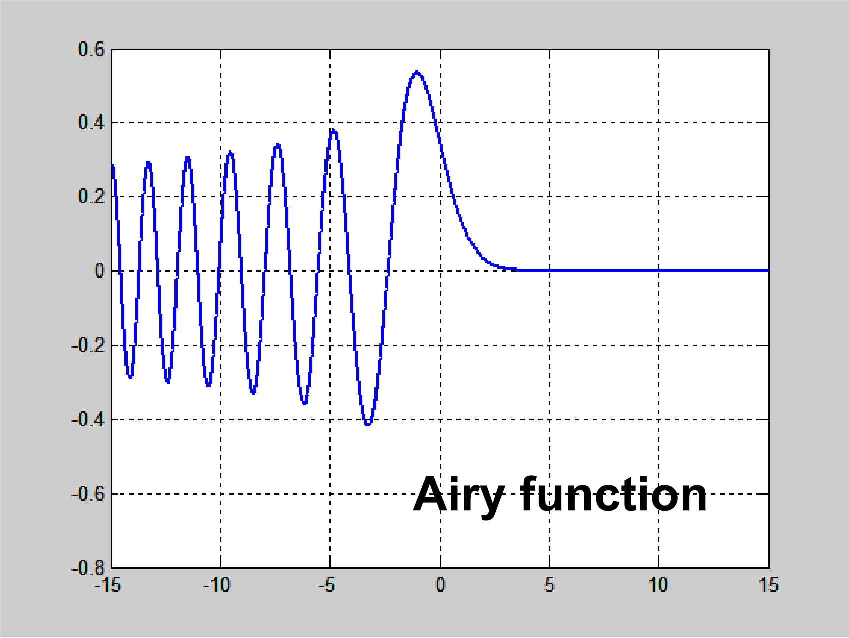




$$< x \,|\, \exp(8 \cdot i \cdot t \cdot \hat{p}^3) \,|\, x' >= \int_{Q(0)=x'}^{Q(t)=x} \exp\left[i \cdot \int_0^t (P(\tau) \cdot \dot{Q}(\tau) + 8 \cdot P^3(\tau)) \cdot d\tau \right] \cdot \prod_\tau \frac{dP(\tau) \cdot dQ(\tau)}{2 \cdot \pi}$$

Green's function of the linearized KdV equation:

$$G(x-x';t)=\int_{-\infty}^{+\infty}\exp[i\cdot p\cdot(x-x')+8\cdot i\cdot t\cdot p^3]\cdot\frac{dp}{2\cdot\pi}=\frac{1}{2\cdot\sqrt[3]{3\cdot t}}\cdot Ai\left(\frac{x-x'}{2\cdot\sqrt[3]{3\cdot t}}\right)$$



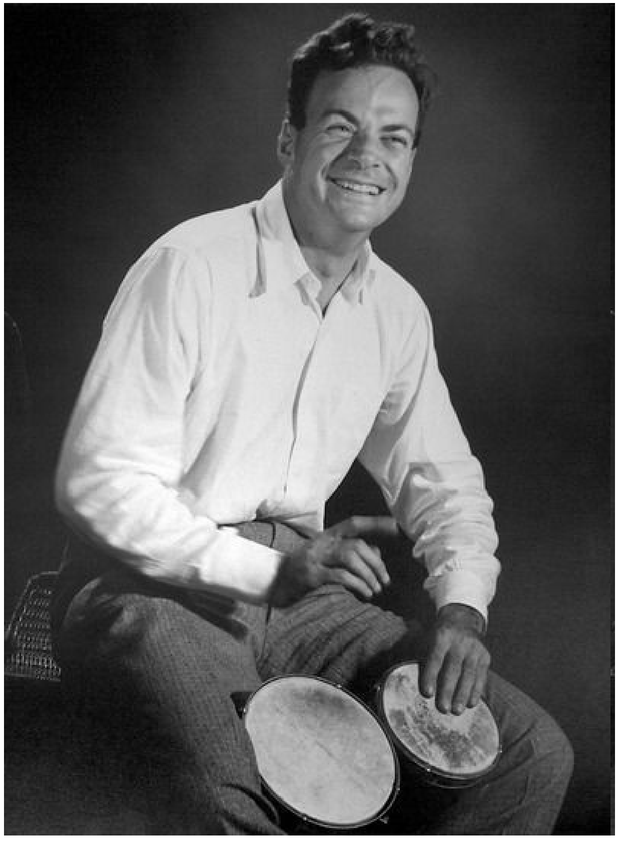
$$Ai(x)=\frac{1}{\pi}\cdot\int_{0}^{+\infty}\cos\left(x\cdot y+\frac{y^3}{3}\right)\cdot dy$$

$$Ai(x)=\frac{1}{\pi}\cdot\sqrt{\frac{x}{3}}\cdot K_{\frac{1}{3}}\left(\frac{2}{3}\cdot x^{\frac{3}{2}}\right),\quad x>0$$

$$Ai(x)=\frac{\sqrt{|x|}}{3}\cdot\left[J_{\frac{1}{3}}\left(\frac{2}{3}\cdot|x|^{\frac{3}{2}}\right)+J_{-\frac{1}{3}}\left(\frac{2}{3}\cdot|x|^{\frac{3}{2}}\right)\right],\quad x<0$$

Definition of feynmanon

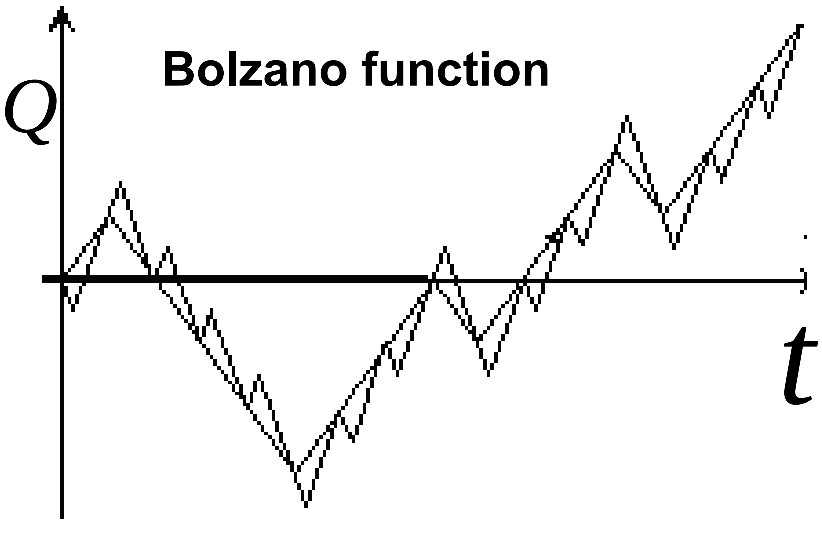
$$\frac{1}{2\cdot\sqrt[3]{3\cdot t}}\cdot Ai\left[\frac{x-x'}{2\cdot\sqrt[3]{3\cdot t}}\right]=\int_{Q(0)=x'}^{Q(t)=x}\exp\left[i\cdot\int_0^t\left(P(\tau)\cdot\dot{Q}(\tau)+8\cdot P^3(\tau)\right)\cdot d\tau\right]\cdot\prod_\tau\frac{dP(\tau)\cdot dQ(\tau)}{2\cdot\pi}$$

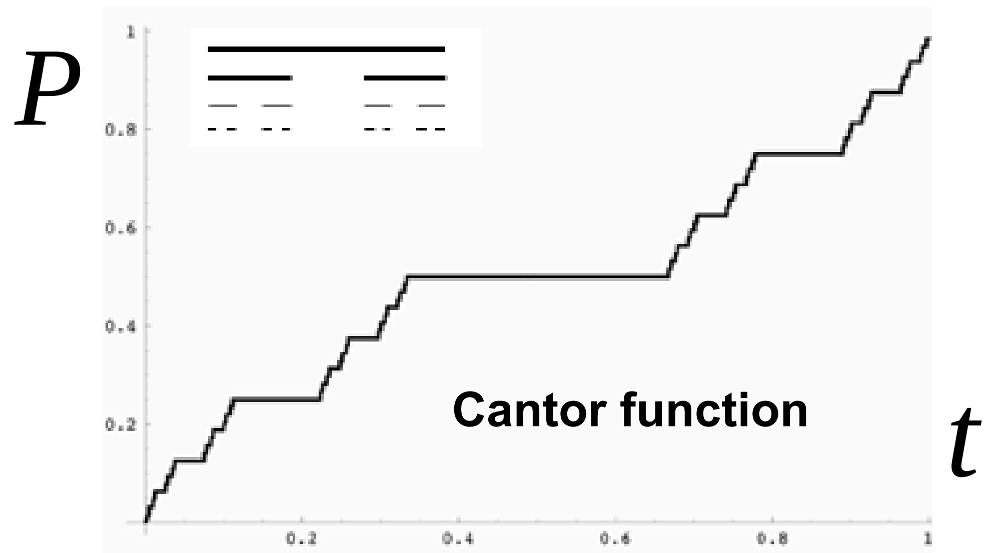


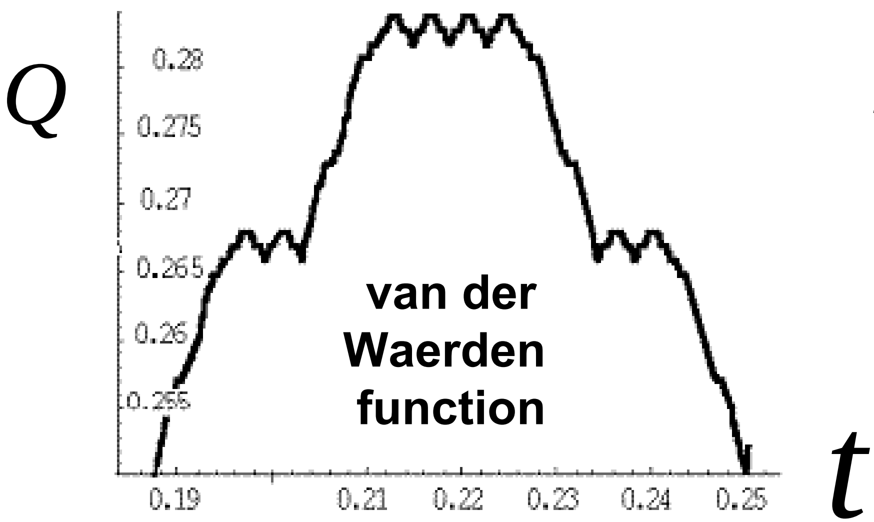
- **Representation of Green's function by Feynman integral gives us the possibility to introduce quantum quasiparticle related with the linearized KdV equation as object moving along nondifferentiable trajectories in phase space (P,Q).**

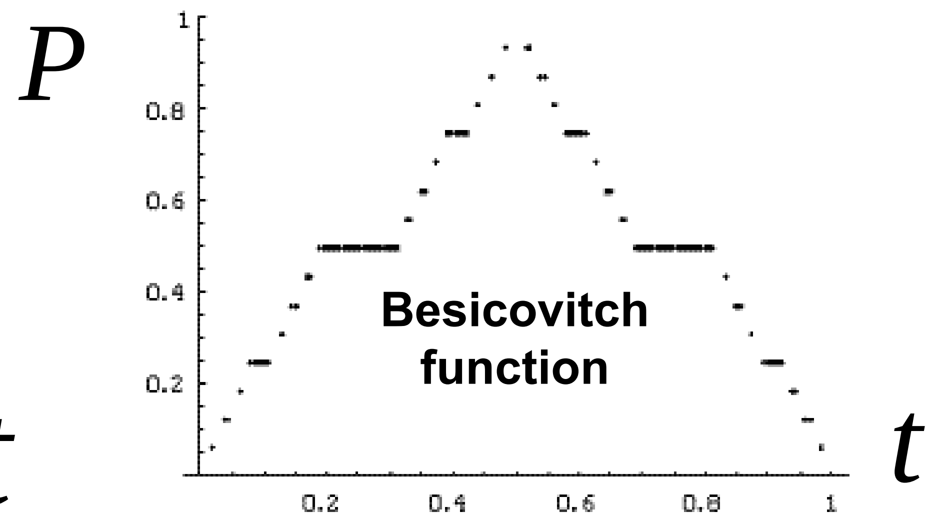
- **In honour of outstanding physicist of the 20th century Richard Feynman (1918-1988) we call this quasiparticle by 'feynmanon'.**

Fractal properties of trajectories in phase space (P,Q): for coordinates (from the left) and for momenta (from the right)



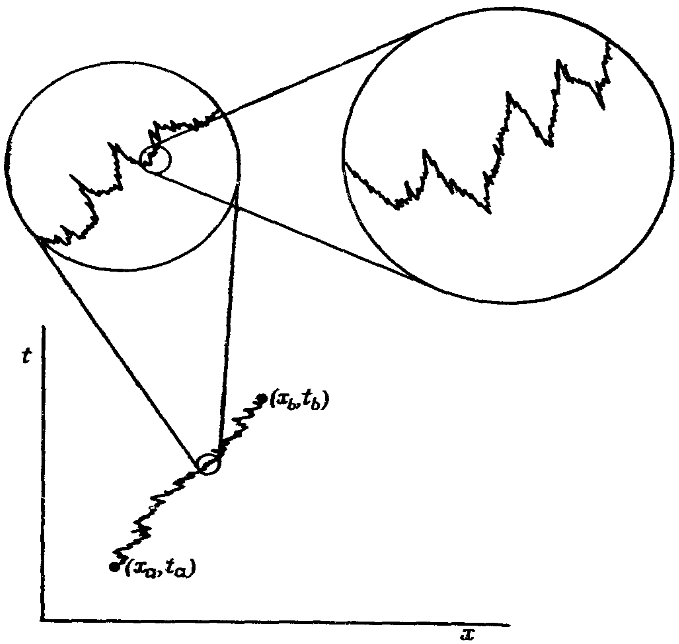






The majority of nondifferentiable trajectories in phase space of Feynman integral is fractal

Banach C. Uber die Baire sche Kategorie gewisser Funktionnen-mengen. Stud. Math.,1931.
Mazurkiewicz S. Sur les fonctions non derivables. Stud. Math., 1931.



$$f(x) = \sum_{n=1}^{\infty} a^n \cdot r(b^n \cdot x)$$

$$0 < a < 1$$

$$a \cdot b > 1$$

$r(x)$ **— quasiperiodic function**

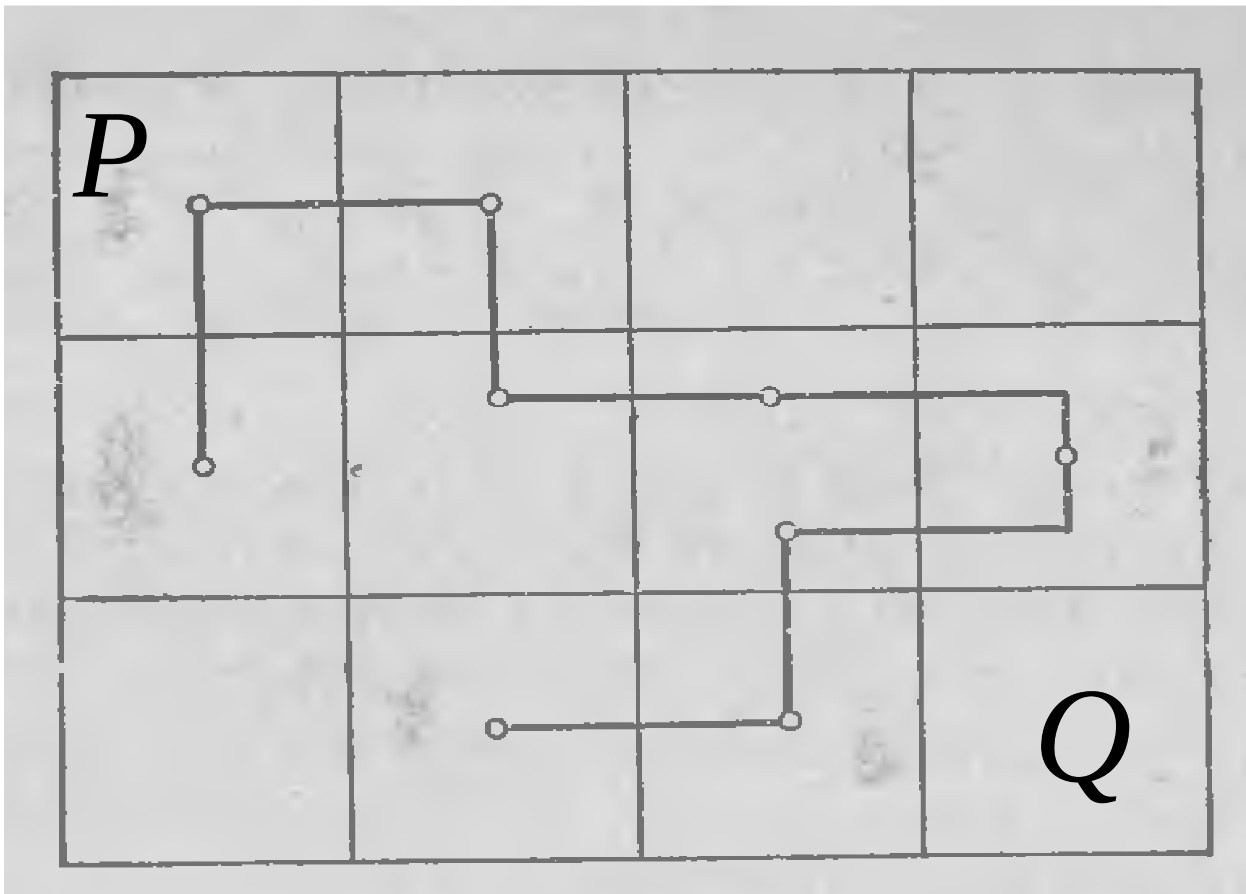
Property of selfsimilarity:

$$f(x) - a \cdot f(b \cdot x) = a \cdot r(b \cdot x)$$

Fractal dimension: $D = 2 + \log_b a$ $\quad 1 < D < 2$

Set of feynmanon's trajectories on plane (P,Q)

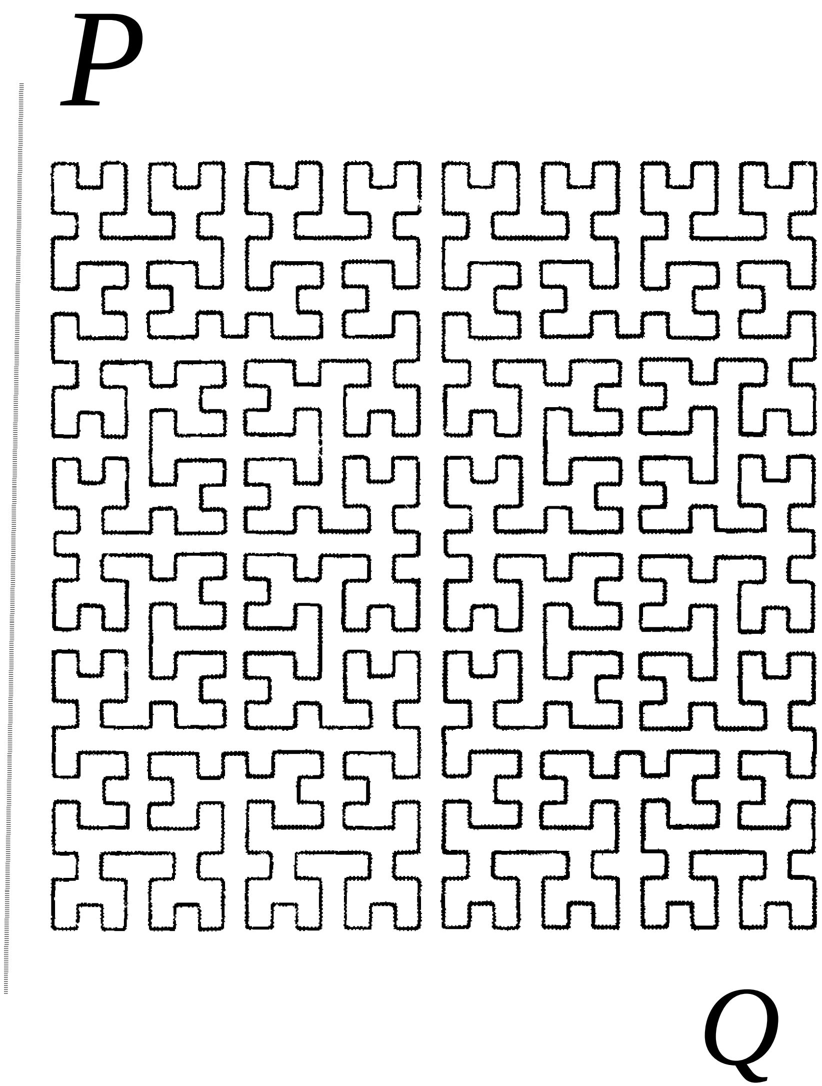
$$Q(\tau) = Q_j + (Q_{j+1} - Q_j)\cdot(\tau - t_j)/\Delta t \quad P(\tau) = P_{j+1} \quad \tau \in [t_j, t_{j+1}]$$

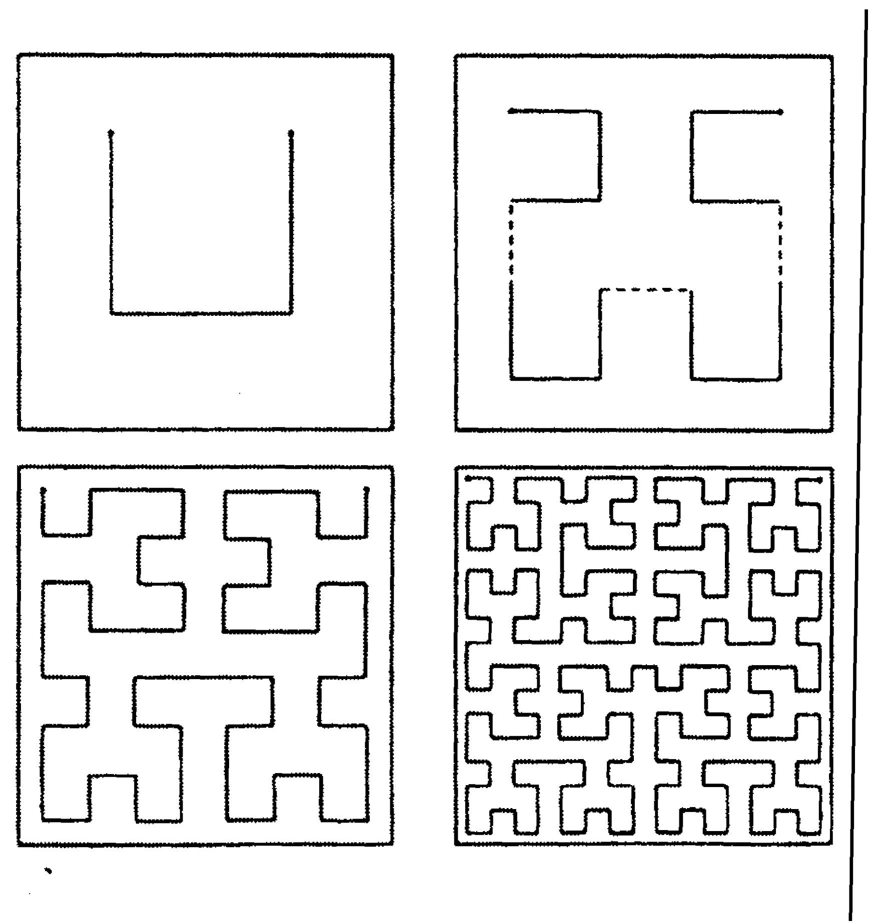


$$j = \overline{0, N}$$

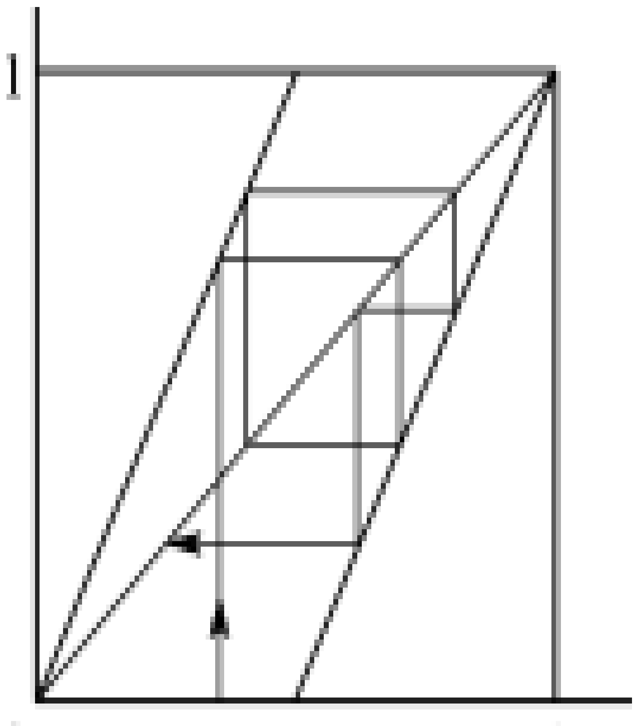
Set of feynmanon's trajectories is two-dimensional random walk

Trajectory of feynmanon: Peano curve





“Instantaneous” dynamics of momenta may obey to succession map:



$$P_{j+1} = f(P_j) \quad j = \overline{1, N}$$

$$P_{j+1} = \{2 \cdot P_j\}$$

This map can have chaotic behaviour!

And invariant measure:

$$\rho(P) = \begin{cases} 1, & P \in [0,1) \\ 0, & P \notin [0,1) \end{cases}$$

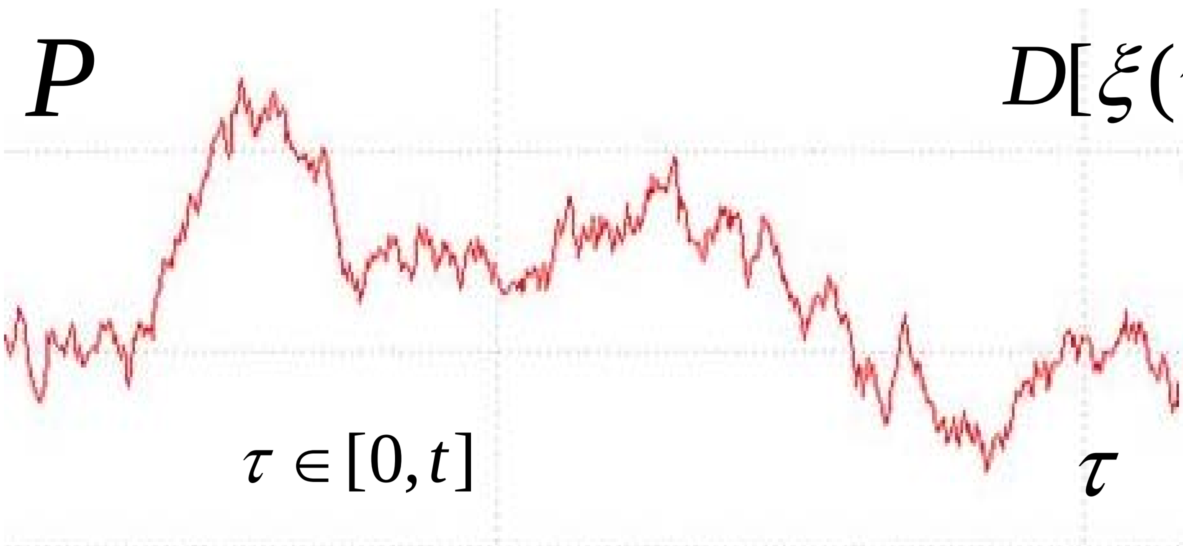
Perron-Frobenius equation:

$$\rho(P) = \sum_k | f'(\xi_k(P)) |^{-1} \cdot \rho(\xi_k(P)) \qquad P = f(\xi(P))$$

Feynman integrals are closely related with dynamical chaos!

Trajectories of momenta as Wiener process

$$P_j = \xi(j \cdot \Delta t) \qquad j = \overline{1, N+1} \qquad M[\xi(\tau)] = 0$$



$$D[\xi(\tau) - \xi(s)] = \tau - s$$

$$MX_n = 0$$

$$\xi(\tau) = \sum_{n=1}^{\infty} X_n \cdot \sin\left[(2 \cdot n - 1) \cdot \frac{\pi \cdot \tau}{2 \cdot t}\right] \qquad DX_n = \frac{4}{\pi \cdot (2 \cdot n - 1)^2}$$

Wiener measure for momenta:

$$W_{\Delta\xi}(P_1,\ldots,P_{N+1}) = \prod_{j=1}^{N+1} \frac{1}{\sqrt{2\cdot\pi\cdot(t_j - t_{j-1})}} \cdot \exp\left[-\frac{P_j^2}{2\cdot(t_j - t_{j-1})}\right]$$

$D = 1{,}5$ **— fractal dimension of Wiener process**

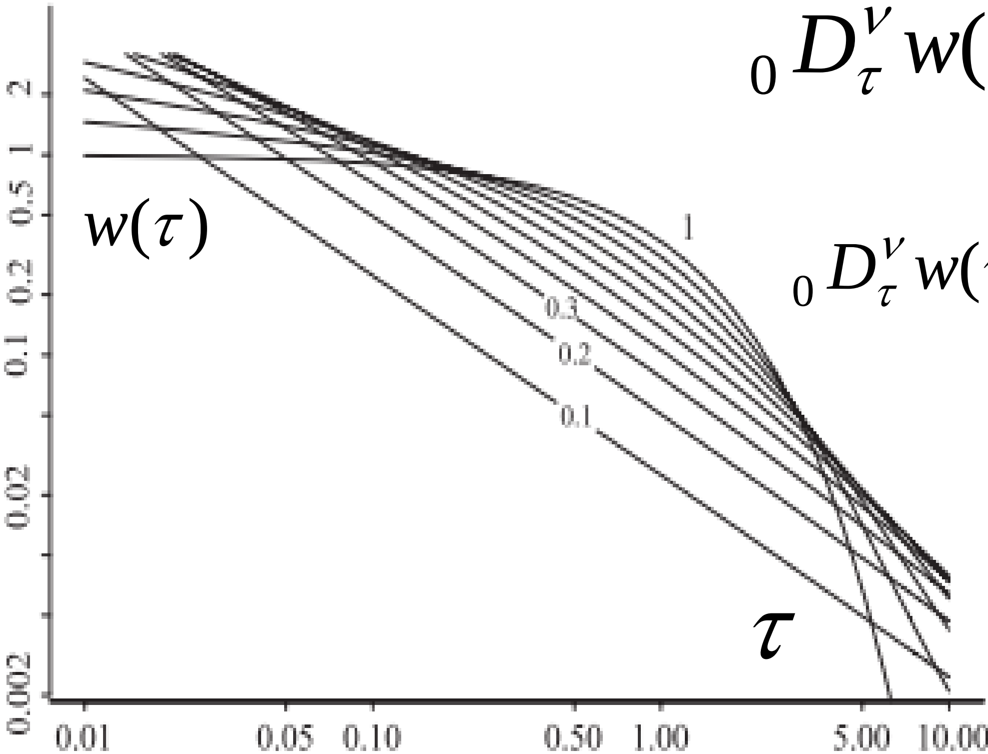
Property of selfsimilarity:

$\xi(s+\Delta s) - \xi(s)$ **— both random values have the same distribution**

$$\frac{1}{\alpha}\cdot[\xi(s+\alpha\cdot\Delta s) - \xi(s)] \qquad \alpha > 0$$

Trajectories of momenta as fractional Poisson process

Repin O. N., Saichev A. I. Izvestiya vuzov. Radiofizika. 43, 823 (2000).



$$_0D_\tau^\nu w(\tau) + \mu \cdot w(\tau) = \mu \cdot \delta(\tau)$$

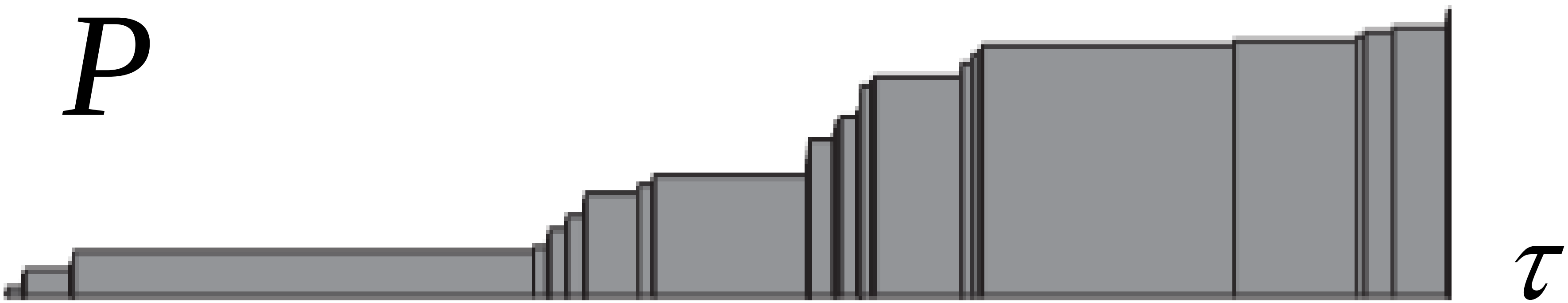
$$_0D_\tau^\nu w(\tau) \equiv \frac{1}{\Gamma(1-\nu)} \cdot \frac{d}{d\tau} \int_0^\tau \frac{w(\theta) \cdot d\theta}{(\tau - \theta)^\nu}$$

$$0 < \nu \leq 1$$

Two-parameter Mittag-Leffler function:

$$w(\tau) = \mu \cdot \tau^{\nu-1} \cdot E_{\nu,\nu}(-\mu \cdot \tau^\nu)$$

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha \cdot n + \beta)}$$



$$< P(\tau) >= \mu \cdot \tau^{\nu} \qquad w(\tau) \approx \frac{\mu^{\nu}}{\Gamma(\nu)} \cdot \tau^{\nu-1} \qquad \tau \to 0+$$

Feynman integrals are closely related with fractional derivatives.

Fractional derivatives are closely related with selfsimilarity.

In the paper O. G. Smolyanov, A. G. Tokarev, A. Truman. Hamiltonian Feynman path integrals via the Chernov formula // J. Math. Phys. V. 43, N 10. p. 5161 (2002) the most rigorous justification of Feynman's pseudomeasure has been done but we see now that submeasures of it can possess by very untrivial properties.

Generalized uncertainty relations for feynmanon's amplitude of probability

$$\int_{-\infty}^{+\infty} |B(x,0)|^2 \, dx = 1 \qquad [-x_0, x_0] \qquad [-p_0, p_0]$$

Let $$\int_{-x_0}^{+x_0} |B(x,0)|^2 \, dx \geq \lambda_0(p_0 \cdot x_0)$$

$\lambda_0(c)$ **— maximal eigenvalue of integral equation:**

$$\int_{-1}^{+1} \frac{\sin[c \cdot (x - x')]}{\pi \cdot (x - x')} \cdot f(x') \cdot dx' = \lambda(c) \cdot f(x)$$

$$\int_{-x_0}^{+x_0} |B(x,t)|^2 \, dx \le \cos^2\left[\arccos\sqrt{\lambda_0(p_0 \cdot x_0)} - \arccos\sqrt{\int_{-p_0}^{+p_0} |b_0(p)|^2 \frac{dp}{2\pi}}\right]$$

$$\frac{\partial |B|^2}{\partial t} + \frac{\partial j}{\partial x} = 0$$ **— equation of continuity**

$$j(x,t) = 8 \cdot \left(B^* \cdot \frac{\partial^2 B}{\partial x^2} + B \cdot \frac{\partial^2 B^*}{\partial x^2} - \frac{\partial B}{\partial x} \cdot \frac{\partial B^*}{\partial x} \right)$$

$$\frac{d}{dt} \int_{-x_0}^{+x_0} |B(x,t)|^2 \, dx = -[\, j(x_0,t) - j(-x_0,t)]$$

The kernel of the GLM equation has another Feynman integral:

$$B(x,t) = \int_{-\infty}^{+\infty} b_0(p) \cdot \exp(i \cdot p \cdot x + 8 \cdot i \cdot p^3 \cdot t) \cdot \frac{dp}{2 \cdot \pi}$$

$$\frac{\exp(i \cdot \vec{k} \cdot \vec{x})}{-2 \cdot \pi \cdot i} = \int_{\vec{Q}(-\pi/2)=\vec{k}}^{\vec{Q}(0)=\vec{x}} \exp\left[-i \cdot \int_{-\pi/2}^{0} \sum_{j=1}^{2} \left(P_j(\tau) \cdot \dot{Q}_j(\tau) - \frac{P_j^2(\tau) + Q_j^2(\tau)}{2} \right) \cdot d\tau \right] \cdot d\mu$$

$$\vec{x} = (x,t)$$

$$\vec{k} = (p, 8 \cdot p^3)$$

$$d\mu = \prod_{j=1}^{2} \prod_{\tau} \frac{dP_j(\tau) \cdot dQ_j(\tau)}{2 \cdot \pi}$$

Further developments:

- **Investigation of effects of multifractality in trajectories of feynmanon;**
- **Extension of methods developed on other nonlinear equations such as Gardner equation, sin-Gordon equation, nonlinear Schrödinger equation, Kadomtzev-Petviashvili equation et all.**

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