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Fractals and path integrals in three-dimensional wave equation

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Three-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \cdot \Delta u$$

$u(\vec{x}, t)$ — **unknown function**

a — **phase velocity**

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

— **Laplacian**

$$\frac{\partial \psi_1}{\partial t} = -a \cdot \sqrt{-\Delta} \cdot \psi_2 \qquad \frac{\partial \psi_2}{\partial t} = a \cdot \sqrt{-\Delta} \cdot \psi_1$$

$$\psi_1(\vec{x}, t) \equiv u(\vec{x}, t) \qquad \psi_2(\vec{x}, t) = -\frac{1}{a} \cdot (-\Delta)^{-\frac{1}{2}} \frac{\partial u(\vec{x}, t)}{\partial t}$$

How to deal with fractional Laplacians?

$$f(\vec{x}) = \int \tilde{f}(\vec{p}) \cdot \exp(i \cdot \vec{p} \cdot \vec{x}) \cdot \frac{d^3 p}{(2 \cdot \pi)^3}$$

$$(-\Delta)^{\alpha} f(\vec{x}) = \int |\vec{p}|^{2\cdot\alpha} \cdot \tilde{f}(\vec{p}) \cdot \exp(i \cdot \vec{p} \cdot \vec{x}) \cdot \frac{d^3 p}{(2 \cdot \pi)^3}$$

$$\sqrt{-\Delta} f(\vec{x}) = -\frac{1}{\pi} \cdot \mathrm{P} \int \frac{f(\vec{x}') \cdot d^3 x'}{|\vec{x} - \vec{x}'|^4}$$

— absolute value of momentum operator

$$\hat{\vec{p}} = -i \cdot \nabla$$

$$(-\Delta)^{-1/2} f(\vec{x}) = \frac{1}{2 \cdot \pi^2} \cdot \int \frac{f(\vec{x}') \cdot d^3 x'}{|\vec{x} - \vec{x}'|^2}$$

Rewrite our system as Schrödinger type equation:

$$i \cdot \frac{\partial |\psi>}{\partial t} = \hat{H} |\psi> \qquad \hat{H} = a \cdot \sqrt{-\Delta} \cdot \sigma_y \qquad |\psi>= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$ **— Pauli matrix** $$\sigma_y^2 = 1$$

$$|\psi(t)>= \exp(-i \cdot t \cdot a \cdot \sqrt{-\Delta} \cdot \sigma_y) \cdot |\psi(0)>$$

$$\psi(\vec{x}, t) = \int \Gamma(\vec{x}, \vec{x}'; t) \cdot \psi(\vec{x}', 0) \cdot d^3 x'$$

$$\Gamma(\vec{x}, \vec{x}'; t) =< \vec{x} \,|\, \exp(-i \cdot t \cdot a \cdot \sqrt{-\Delta} \cdot \sigma_y) \,|\, \vec{x}' >$$ **— Green's matrix**

$$\exp(-i\cdot t\cdot a\cdot\sqrt{-\Delta}\cdot\sigma_y)=\frac{1-\sigma_y}{2}\cdot\exp(i\cdot t\cdot a\cdot\sqrt{-\Delta})+\frac{1+\sigma_y}{2}\cdot\exp(-i\cdot t\cdot a\cdot\sqrt{-\Delta})$$

$$\Gamma(\vec{x},\vec{x}';t)=\frac{1-\sigma_y}{2}\cdot G^*(\vec{x}',\vec{x};t)+\frac{1+\sigma_y}{2}\cdot G(\vec{x},\vec{x}';t)$$

Green's function:

$$G(\vec{x},\vec{x}';t)=<\vec{x}\,|\exp(-i\cdot t\cdot a\cdot\sqrt{-\Delta})|\,\vec{x}'>$$

$$G(\vec{x},\vec{x}';t)=\frac{i}{2\cdot\pi^2\cdot a}\cdot\frac{\partial}{\partial t}\frac{1}{(\vec{x}-\vec{x}')^2-(a\cdot t)^2+i\cdot 0}$$

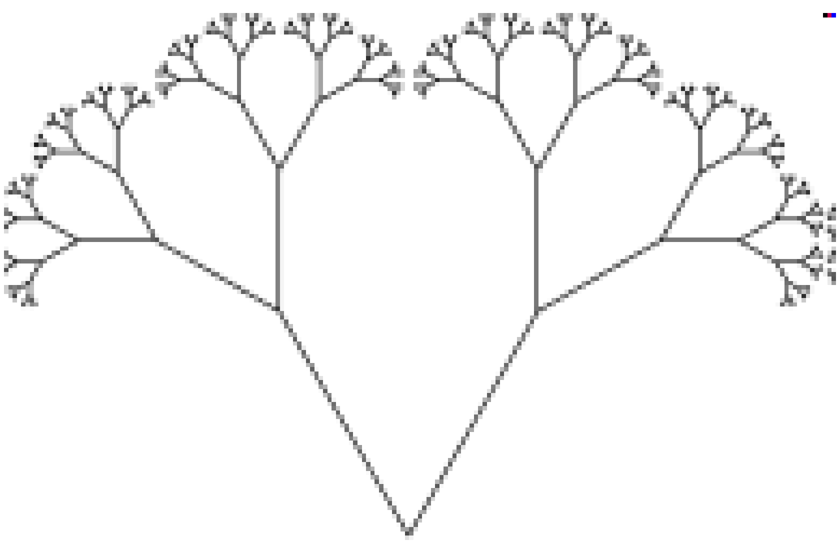
Green's function as Feynman integral:

$$G(\vec{x},\vec{x}';t)=\int_{\vec{Q}(0)=\vec{x}'}^{\vec{Q}(t)=\vec{x}}\exp[i\cdot\int_0^t(\vec{P}(\tau)\cdot\dot{\vec{Q}}(\tau)-a\cdot|\vec{P}(\tau)|)\cdot d\tau]\cdot\prod_{\tau}\frac{d\vec{P}(\tau)\cdot d\vec{Q}(\tau)}{2\cdot\pi}$$

Properties of feynmanon's trajectories in 6D phase space

$$\vec{Q}(\tau)=\vec{Q}_j+(\vec{Q}_{j+1}-\vec{Q}_j)\cdot(\tau-\tau_j)/\Delta\tau\,,\quad \vec{P}(\tau)=\vec{P}_{j+1},\quad \tau\in[\tau_j,\tau_{j+1}]$$

$$\tau_j = j\cdot\Delta\tau \qquad \Delta\tau=\frac{t}{N+1} \qquad j=\overline{0,N}$$



Set of feynmanon's trajectories in 6D phase space is six-dimensional random walk. In particular along 6D Peano curves.

Dynamics of momenta obeys to succession map:

$$\vec{P}_{j+1}=\vec{F}(\vec{P}_j)$$

Coordinates may walk on fractal trees in R^3

Williams-Hatchinson theorem

$$\vec{f}_k : R^d \to R^d$$ — **contraction maps** $$k = \overline{1, s}$$

$$|\vec{f}_k(\vec{x}) - \vec{f}_k(\vec{y})| \le \lambda_k \cdot |\vec{x} - \vec{y}| \qquad 0 < \lambda_k < 1$$

$$D_0 \subset R^d$$ — **compact domain**

$$D_{n+1} = \vec{f}_1(D_n) \bigcup \vec{f}_2(D_n) \bigcup \ldots \bigcup \vec{f}_s(D_n)$$

The limit exist:

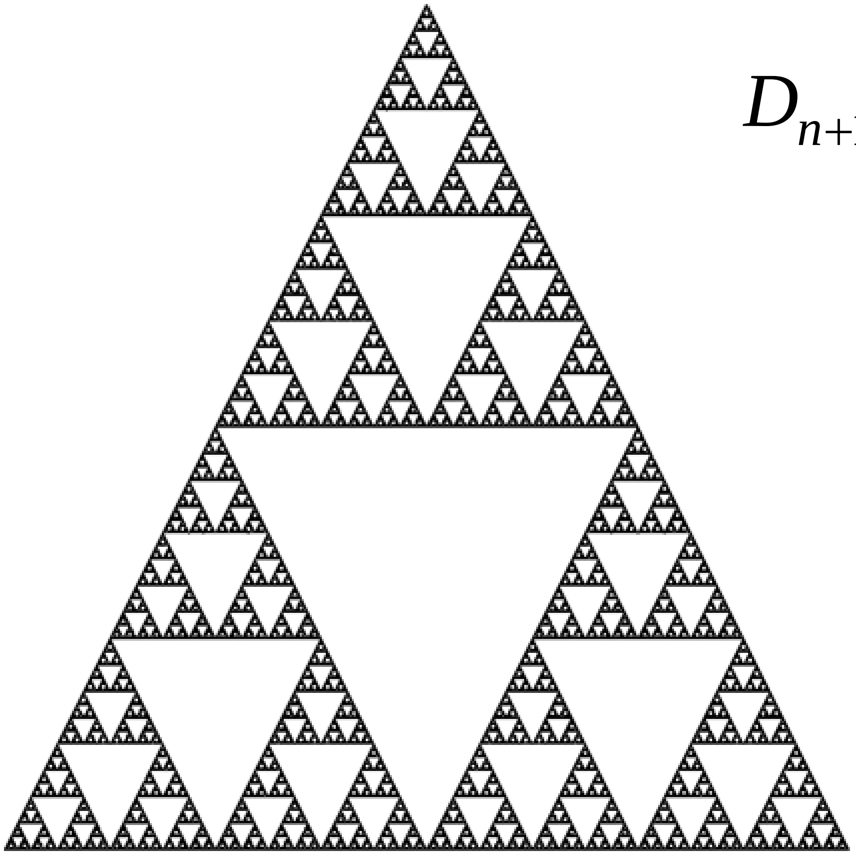
$$D = \lim_{n \to +\infty} D_n$$

Convergence in Hausdorff metric:

$$\rho_H(U, V) = \max\{\max_{\forall \vec{x} \in U} \min_{\forall \vec{y} \in V} |\vec{x} - \vec{y}|, \quad \max_{\forall \vec{y} \in V} \min_{\forall \vec{x} \in U} |\vec{x} - \vec{y}|\}$$

The example of construction of fractal fractal set on two-dimensional plane by means of application of Williams-Hatchinson theorem

$$f_i(\vec{x}) = \frac{1}{2}\cdot\vec{x} + \vec{b}_i \qquad \vec{b}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{b}_2 = \frac{1}{2}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{b}_3 = \frac{1}{4}\cdot\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$



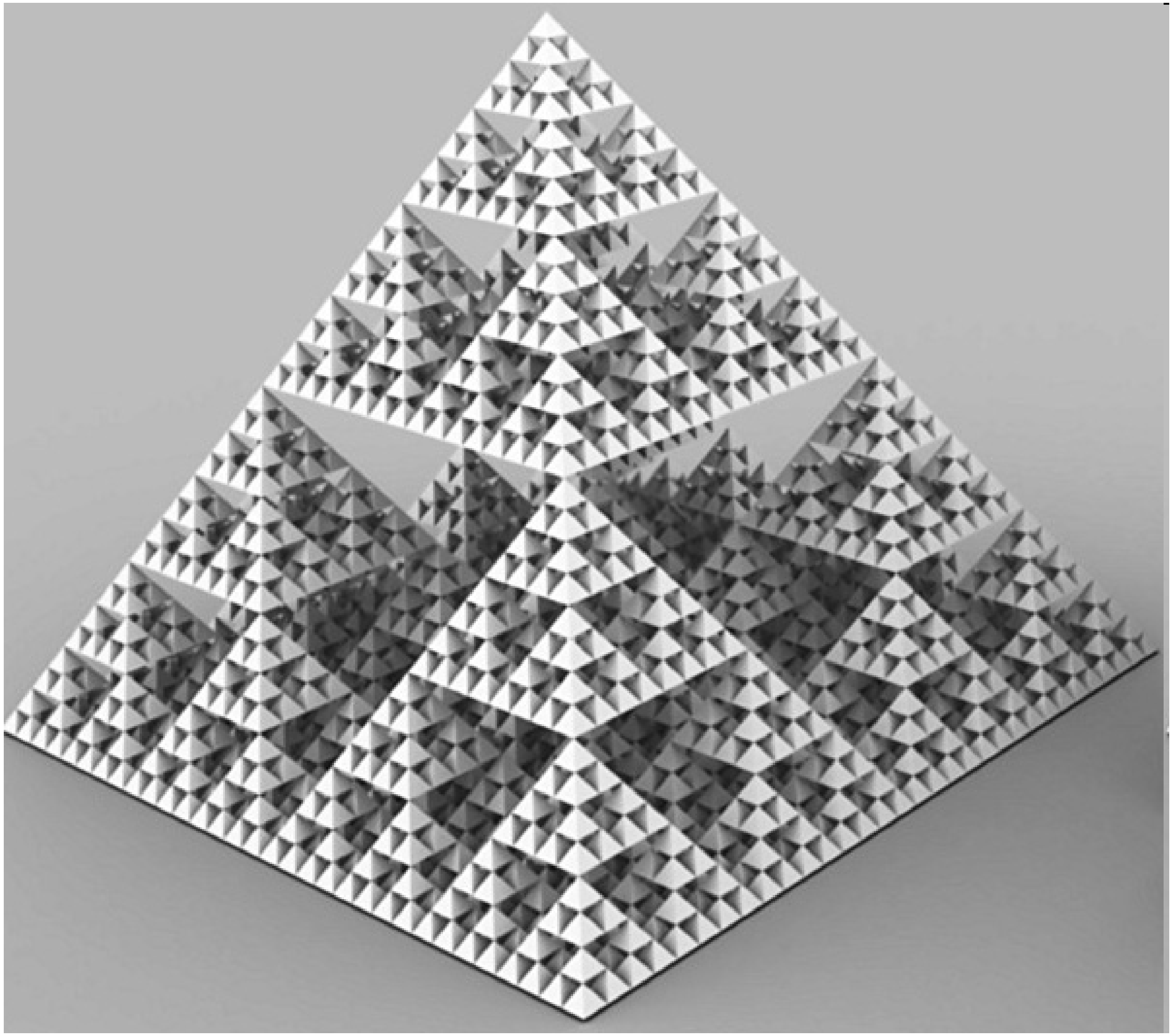
$$D_{n+1} = f_1(D_n)\bigcup f_2(D_n)\bigcup f_3(D_n)$$

$$\lim_{n\to+\infty} \rho_H(D_n, D) = 0$$

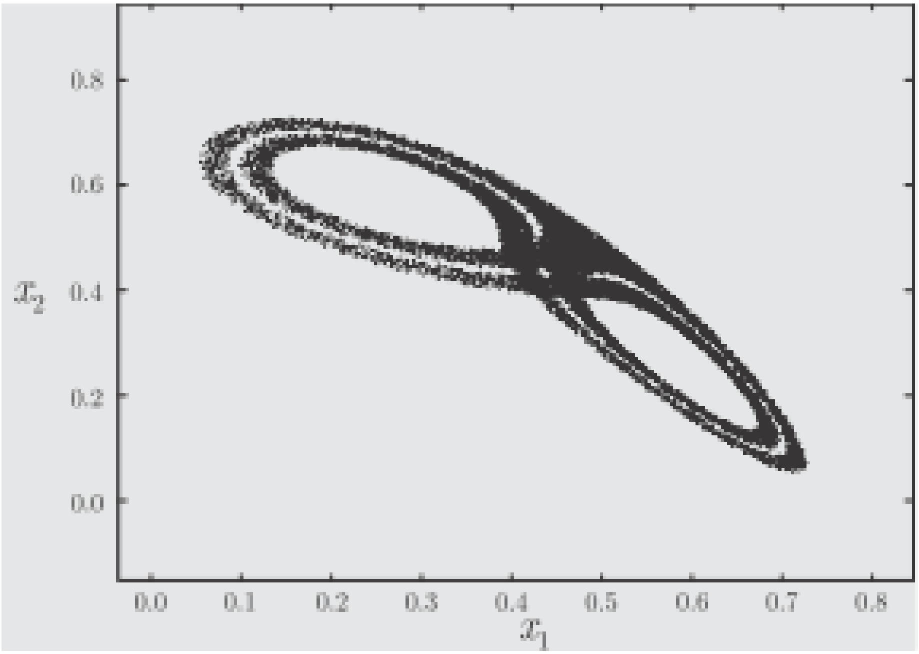
D is Sierpinski napkin. Its fractal dimension is equal to:

$$Dfr = \frac{\ln 3}{\ln 2} \approx 1{,}5849$$

The result of application of Williams-Hatchinson theorem in three-dimensional space



In map for momenta also may take place chaotic behaviour



Gonchenko A.S., Gonchenko S. V., Shilnikov L. P. Towards scenarios of chaos appearance in three-dimensional maps // Rus. J. Nonlin. Dyn., 2012, vol. 8, no. 1, pp. 3–28 (Russian).

$$\overline{P_1} = P_2 \quad \overline{P_2} = P_3$$

$$\overline{P_3} = -0{,}045 + 0{,}85 \cdot P_2 + 0{,}7 \cdot P_1 - P_3^2$$

Quantizing of massless scalar field

$$\hat{u}(\vec{x},t) = \int [\hat{c}(\vec{p}) \cdot \exp(i \cdot \vec{p} \cdot \vec{x} - i \cdot a \cdot |\vec{p}| \cdot t) + h.c.] \cdot \frac{d^3 p}{\sqrt{2 \cdot a \cdot |\vec{p}|} \cdot (2 \cdot \pi)^{3/2}}$$

Bose canonical commutative relations:

$$[\hat{c}(\vec{p}), \hat{c}^+(\vec{p}\,')] = \delta(\vec{p} - \vec{p}\,') \qquad [\hat{c}(\vec{p}), \hat{c}(\vec{p}\,')] = 0$$

$$\frac{\partial^2 \hat{u}}{\partial t^2} = a^2 \cdot \Delta \hat{u}$$

Hamiltonian of massless scalar field:

$$\hat{H} = \int : \left[\frac{1}{2} \cdot \left(\frac{\partial \hat{u}(\vec{x},t)}{\partial t} \right)^2 + \frac{a^2}{2} \cdot (\nabla \hat{u}(\vec{x},t))^2 \right] : d^3 x$$

$$\hat{H} = \int a \cdot |\vec{p}| \cdot \hat{c}^{+}(\vec{p}) \cdot \hat{c}(\vec{p}) \cdot d^{3}p$$

$$\hat{H} = \int : \left[\frac{1}{2} \cdot \left(\frac{\partial \hat{u}(\vec{x},t)}{\partial t} \right)^{2} + \frac{a^{2}}{2} \cdot (\sqrt{-\Delta} \cdot \hat{u}(\vec{x},t))^{2} \right] : d^{3}x$$

The result is exactly the same!

THANK YOU FOR YOUR ATTENTION!