# Global well-posedness and attractors for the hyperbolic Cahn-Hilliard-Oono equation in $\mathbb{R}^3$


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# Outline

# Statement of the problem

We consider the so called hyperbolic Cahn-Hilliard-Oono equation

$$\varepsilon\partial_t^2 u + \partial_t u + \alpha u + \Delta_x(\Delta_x u - f(u) + g) = 0,\ x \in \Omega, \quad (1)$$

$$(u, \partial_t u)|_{t=0} = (u_0, u_0') \in \mathcal{E} := \left(\dot{H}^1 \cap \dot{H}^{-1}\right) \times \dot{H}^{-1} \quad (2)$$

where $\Omega = \mathbb{R}^3$ and

1. $\varepsilon = 1$ and $\alpha > 0$;
2. $g = g(x) \in \dot{H}^1$;
3. $f \in C^2(\mathbb{R})$ such that

$$f(u)u \geqslant 0, \quad (3)$$

$$F(u) \leqslant Lf(u)u + K|u|^2,\ F(u) = \int_0^u f(v)dv,\ L,\ K \geqslant 0, \quad (4)$$

$$|f''(u)| \leqslant C(1 + |u|^q), \quad q \in [0, 3). \quad (5)$$

# Overview: parabolic case

$\alpha = 0$, J. Cahn, J. Hilliard 1958:

$$\partial_t u + \alpha u + \Delta_x (\Delta_x u - f(u) + g) = 0, \ x \in \Omega.$$

$\Omega$ - bdd: well understood, see e. g.

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Restoring dissipation: Y. Oono, S. Puri 1987 - $\alpha > 0$

1. J. Pennant, S. Zelik 2013;

# Overview: hyperbolic case, $\varepsilon > 0$

P. Galenko 2001, 2005, 2008: hyperbolic CHO equation with $\alpha = 0$.

$$\varepsilon\partial_t^2 u + \partial_t u + \alpha u + \Delta_x(\Delta_x u - f(u) + g) = 0, \ x \in \mathbb{R}^3.$$

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3. $\Omega \subset \mathbb{R}^3$ and bdd, Dirichlet BC:
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   No uniqueness is known in $L^\infty([0, T]; H_0^1(\Omega) \times H^{-1}(\Omega))$ even for bdd $f$.

# Overview: hyperbolic case, $\varepsilon > 0$

## Notations

$(\cdot,\cdot)$ - *scalar product in* $L^2$, $\|\cdot\|$-*norm in* $L^2$;
$\xi_u := (u, \partial_t u)$, $\|\xi_u(t)\|^2_{\mathcal{E}} = \|u(t)\|^2_{\dot{H}^1} + \alpha\|u(t)\|^2_{\dot{H}^{-1}} + \|\partial_t u(t)\|^2_{\dot{H}^{-1}}$.

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## Energy identity

$$\frac{d}{dt}\left(\frac{1}{2}\|\xi_u\|^2_{\mathcal{E}} + (F(u),1) - (g,u)\right) + \|\partial_t u\|^2_{\dot{H}^{-1}} = 0.$$

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## Definition 1

*Function* $u$ *such that* $\xi_u(t) \in L^\infty([0,T];\mathcal{E})$ *to be called* **energy solution** *of* (1)-(2) *on* $[0,T]$ *if it solves the problem in the sense of distributions and* $\xi_u(0) = (u_0, u_0')$.

# Overview: hyperbolic case, $\varepsilon > 0$

**Theorem 1 (Existence of Energy Solutions and Dissipative Estimate)**

*Let $f$ satisfies dissipative and growth assumptions, then problem* (1)-(2) *possesses global energy solution $u$ such that* $\xi_u(t) \in L^\infty(\mathbb{R}^+;\mathcal{E})$ *and for all* $t \geqslant 0$

$$\|\xi_u(t)\|_{\mathcal{E}}^2 + \int_t^{t+1} \|\partial_t u(s)\|_{\dot{H}^{-1}}^2 ds \leqslant Q(\|\xi_u(0)\|_{\mathcal{E}})e^{-\beta t} + Q(\|g\|_{\dot{H}^1}).$$

*where $\beta > 0$ and $Q$ is a monotone increasing function which is independent of $t$ and $u$.*

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*where $\beta > 0$ and $Q$ is a monotone increasing function which is independent of $t$ and $u$.*

3D: no uniqueness is known even for bdd $f$.

## Plate Equation and Variation of Constants Formula

$$\begin{cases} \partial_t^2 V + \Delta_x^2 V = \Delta_x H(t), \ H(t) \in L^1_{loc}(\mathbb{R}; \dot{H}^1), \\ V|_{t=0} = V_0 \in \dot{H}^1, \ \partial_t V|_{t=0} = V_1 \in \dot{H}^{-1}. \end{cases}$$

$$V(t) = \sin(\Delta_x t)(\Delta_x)^{-1} V_1 + \cos(\Delta_x t) V_0 + \int_0^t \sin(\Delta_x (t-s)) H(s) ds.$$

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## Theorem 2 (J. Ginibre, G. Velo 1985; T. Cazenave 2003)

$$\begin{cases}\partial_t U - i\Delta_x U = H(t), \ H(t) \in L^1_{loc}(\mathbb{R}; \dot{H}^1), \\ U|_{t=0} = U_0 \in \dot{H}^1.\end{cases}$$

*Then* $U \in C(\mathbb{R}; \dot{H}^1) \bigcap L^4_{loc}(\mathbb{R}; L^\infty)$ *and*

$$\|U\|_{C([-T,T];\dot{H}^1)} + \|U\|_{L^4([-T,T];L^\infty)} \leqslant C_T \left( \|U_0\|_{\dot{H}^1} + \|H\|_{L^1([-T,T];\dot{H}^1)} \right).$$

## Definition 2 (S-solution)

*Energy solution* $u(t)$ *of problem* (1)-(2) *is called* $S-$ *solution iff it possesses the extra regularity* $\xi_u \in C([0,T];\mathcal{E})$ *and* $u \in L^4([0,T];C_b)$.

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## Theorem 3 (Existence of S-solutions and dissipative estimate)

*Let $f$ be of subquintic growth and satisfies dissipative assumption. Then problem (1)-(2) possesses global S-solution $u$ such that $\xi_u(t) \in C_b(\mathbb{R}_+;\mathcal{E})$ and $u \in L^4_{loc}(\mathbb{R}_+;C_b)$ and dissipative estimate holds*

$$\|\xi_u(t)\|^2_{\mathcal{E}} + \int_t^{t+1} \|\partial_t u(s)\|^2_{\dot{H}^{-1}} ds + \|u\|^2_{L^4([t,t+1];C_b)} \leqslant Q(\|\xi_u(0)\|_{\mathcal{E}})e^{-\beta t} + Q(\|g\|_{\dot{H}^1}), \ t \geqslant 0.$$

*where $\beta > 0$ and $Q$ is a monotone increasing function which is independent of $t$ and $u$.*

# Uniqueness and continuous dependence $S$-solutions

**Corollary**

1. *$S$-solutions obey energy identity, indeed $f(u) \in L^1_{loc}(\mathbb{R}_+; H^1)$;*
2. *$S$-solutions are unique. Moreover,*

$$\|\xi_{u_1}(t) - \xi_{u_2}(t)\|_{\mathcal{E}} + \|u_1 - u_2\|_{L^4([t,t+1];C_b)} \leqslant Ce^{Kt}\|\xi_{u_1}(0) - \xi_{u_2}(0)\|_{\mathcal{E}},$$

*where $C$ and $K$ depend on $\|\xi_{u_1}(0)\|_{\mathcal{E}}$, $\|\xi_{u_2}(0)\|_{\mathcal{E}}$ only.*

# Basic definitions

Let $\mathcal{E}$ be a complete metric space and $S_t : \mathcal{E} \to \mathcal{E}$ be a semigroup on $\mathcal{E}$.

## Definition 3

*A subset $A \subset \mathcal{E}$ is called a* **global attractor** *for the dynamical system $(\mathcal{E}, S_t)$, if*

1. *$A$ is compact in $\mathcal{E}$;*
2. *$A$ is invariant, i.e. $S_t A = A \ \forall t \geqslant 0$;*
3. *for any bounded set $B \subset \mathcal{E}$*
   $\lim_{t\to\infty} \sup\{dist(S_t y, A) : \ y \in B\} = 0.$

# Main result

## Theorem 4 (A. S., S. Zelik)

*Let $f$ be of subquintic growth and satisfies dissipative assumptions. Then the $S$-solution semigroup $S(t)$ associated with problem* (1)-(2)*:*

$$S_t : \mathcal{E} \to \mathcal{E}, \quad S_t\xi_0 = \xi_u(t),\ \xi_u(0) = \xi_0,$$

*possesses a global attractor $\mathcal{A}$ in $\mathcal{E}$. Furthermore,*

$$\|\mathcal{A}\|_{\mathcal{E}_2} \leqslant C_{\mathcal{A}},$$

*where $\mathcal{E}_2 := \dot{H}^3 \cap \dot{H}^{-1} \times \dot{H}^1 \cap \dot{H}^{-1}$.*

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**Definition 4 (Absorbing set)**

*A set $D$ to be called* **absorbing** *for the dynamical system $(\mathcal{E}, S_t)$ iff:*
$\forall$ *bdd* $B \subset \mathcal{E}\ \exists T = T(B) : \ \forall t \geqslant T\ S_t B \subset D.$

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**Definition 5 (Asymptotic compactness)**

*The $(\mathcal{E}, S_t)$ is* **asymptotically compact** *iff for any bdd set $B \subset \mathcal{E}$, any sequence of the initial data $\xi_n \in B$ and any sequence of times $t_n \geq 0$ such that $t_n \to +\infty$ as $n \to \infty$, the sequence*

$$\{S_{t_n}\xi_n\}_{n=1}^{\infty} \text{ is precompact in } \mathcal{E}.$$

# Classic result

R. Temam 1988; A. Babin, M. Vishik 1990.

**Theorem 5 (Existence of the attractor)**

*Let the semi-group $S_t : \mathcal{E} \to \mathcal{E}$ possess the following properties:*

1. *The operators $S_t : \mathcal{E} \to \mathcal{E}$ are continuous in $\mathcal{E}$ for every fixed $t$;*
2. *$(\mathcal{E}, S_t)$ possesses a bounded absorbing set (= is dissipative);*
3. *$(\mathcal{E}, S_t)$ is asymptotically compact.*

*Then dynamical system $(\mathcal{E}, S_t)$ possesses a global attractor $\mathcal{A} \subset \mathcal{E}$, which is generated by all complete trajectories of the semi-group $S_t$:*

$$\mathcal{A} = \mathcal{K}\big|_{t=0},$$

*where $\mathcal{K} \subset L^\infty(\mathbb{R}, \mathcal{E})$ consists of all bounded functions $u : \mathbb{R} \to \mathcal{E}$ such that $S_h u(t) = u(t+h)$ for all $t \in \mathbb{R}$ and $h \geq 0$.*

# Scheme of the proof: asymptotic compactness

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$$\begin{cases} \partial_t^2 u_n + \partial_t u_n + \alpha u_n + \Delta_x \left(\Delta_x u_n - f(u_n) + g\right) = 0, \\ (u_n, \partial_t u_n)|_{t=-t_n} = \xi_n. \end{cases} \tag{6}$$

$$S_{t_n}\xi_n = \xi_{u_n}(0)\ \text{ -- } \textit{bdd by dissipativity}. \tag{7}$$

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3.

$$\xi_{u_n}(0) \rightharpoonup \xi_u(0)\ \textit{a. e. wrt } x,\ \textit{weakly in } \mathcal{E},\ \textit{where} \tag{8}$$

$u(t) \in C_b(\mathbb{R}, \mathcal{E}) \cap L^4_{loc}(\mathbb{R}, C_b)$ is a complete S-solution.

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4. To prove

$$\xi_{u_n}(0) \to \xi_u(0)\ \textit{strongly}\ in\ \mathcal{E} \tag{9}$$

It is enough

$$\|\xi_{u_n}(0)\|_{\mathcal{E}} \to \|\xi_u(0)\|_{\mathcal{E}},\ n \to \infty. \tag{10}$$

5. Energy method (R. Rosa et al. 1998, J. Ball 2004)

$$\bar{E}_u(t) :=$$

$$\frac{1}{4}\|\xi_u(t)\|^2_{\mathcal{E}}+\delta(\partial_t u(t),(-\Delta_x)^{-1}u(t))+\frac{\delta}{2}\|u(t)\|^2_{\dot{H}^{-1}}+(F(u(t)),1),$$

$\delta > 0$ - small enough.

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$\delta > 0$ - small enough.

$$\frac{d}{dt}\left(\frac{1}{4}\|\xi_{u_n}(t)\|^2_{\mathcal{E}}+\bar{E}_{u_n}(t)\right)+\beta\left(\frac{1}{4}\|\xi_{u_n}(t)\|^2_{\mathcal{E}}+\bar{E}_{u_n}(t)\right)=-\mathcal{H}_{u_n}(t),$$

$\beta > 0$ - small enough.

$$\frac{1}{4}\|\xi_{u_n}(0)\|^2_{\mathcal{E}}+\bar{E}_{u_n}(0)=$$

$$e^{-\beta t_n}\left(\frac{1}{4}\|\xi_{u_n}(-t_n)\|^2_{\mathcal{E}}+\bar{E}_{u_n}(-t_n)\right)-\int_{-t_n}^{0}e^{\beta t}\mathcal{H}_{u_n}(t)dt.$$

By weak convergence and Fatou lemma

$$\bar{E}_u(0) \leqslant \varliminf_{n\to\infty} \bar{E}_{u_n}(0), \ \int_{-\infty}^{0} e^{\beta s}\mathcal{H}_u(s)ds \leqslant \varliminf_{n\to\infty} \int_{-t_n}^{0} e^{\beta s}\mathcal{H}_{u_n}(s)ds.$$

$$\frac{1}{4}\varlimsup_{n\to\infty} \|\xi_{u_n}(0)\|_{\mathcal{E}}^2 + \bar{E}_u(0) \leqslant -\int_{-\infty}^{0} e^{\beta s}\mathcal{H}_u(s)ds. \qquad (11)$$

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However, since u obeys energy identity,

$$\frac{1}{4}\|\xi_u(0)\|^2_{\mathcal{E}} + \bar{E}_u(0) = -\int_{-\infty}^{0} e^{\beta s}\mathcal{H}_u(s)ds. \tag{12}$$

Thus

$$\varlimsup_{n\to\infty} \|\xi_{u_n}(0)\|^2_{\mathcal{E}} \leqslant \|\xi_u(0)\|^2_{\mathcal{E}} \leqslant \varliminf_{n\to\infty} \|\xi_{u_n}(0)\|^2_{\mathcal{E}}.$$

6. Smoothness of the attractor follows from its compactness.

# Finite fractal dimension of the attractor

## Definition 6

**Fractal dimension** *$dim_f^{\mathcal{E}} M$ of a compact set $M \subset \mathcal{E}$ in a metric space $\mathcal{E}$ is defined by*

$$dim_f^{\mathcal{E}} M = \overline{\lim_{\varepsilon \to 0}} \frac{\log n(M, \varepsilon)}{\log\left(\frac{1}{\varepsilon}\right)},$$

*where $n(M, \varepsilon)$ is the minimal number of closed $\varepsilon$-balls that cover $M$.*

## Theorem 6 (A. S., S. Zelik)

*Let $f$ be of subquintic growth and satisfies dissipative assumptions. And let $\mathcal{A}$ be the global attractor of the dynamical system associated with S-solutions of the hyperbolic CHO equation. Then*

$$dim_f^{\mathcal{E}} \mathcal{A} < \infty.$$

# Basics of multilinear algebra

Let $\mathcal{E}$ be a separable Hilbert space.

## Definition 7 (Wedge-product)

*Let $\phi_1, \ldots, \phi_d \in \mathcal{E}$, then $\phi_1 \wedge \ldots \wedge \phi_d$ is a $d$-linear antisymmetric form on $\mathcal{E}$ defined by:*

$$\phi_1 \wedge \ldots \wedge \phi_d(\psi_1, \ldots, \psi_d) := det\left((\phi_i, \psi_j)_{i,j=1}^d\right), \quad \psi_1, \ldots, \psi_d \in \mathcal{E}.$$

$\tilde{\Lambda}^d \mathcal{E}$ - space of all finite $d$-linear antisymmetric functionals that can be represented as finite linear combination of terms $\phi_1 \wedge \ldots \wedge \phi_d$.

## Definition 8 (Scalar product)

$$(\phi_1 \wedge \ldots \wedge \phi_d, \psi_1 \wedge \ldots \wedge \psi_d)_{\Lambda^d \mathcal{E}} = det\left((\phi_i, \psi_j)_{i,j=1}^d\right)$$

## Definition 9

*$\Lambda^d \mathcal{E}$- completion of $\tilde{\Lambda}^d \mathcal{E}$ with respect to $\| \cdot \|_{\Lambda^d \mathcal{E}}$.*

# Volume contraction factor

**Definition 10 ($\Lambda^d L$)**

*Let $L \in \mathcal{L}(\mathcal{E}, \mathcal{E})$. Then the linear operator $\Lambda^d L$ acts on $\Lambda^d \mathcal{E}$ by:*
$(\Lambda^d L)\xi(\psi_1, \ldots, \psi_d) = \xi(L^*\psi_1, \ldots, L^*\psi_d)$,
*where $L^* \in \mathcal{L}(\mathcal{E}, \mathcal{E})$ is the adjoint to $L$.*

In particular, if $\xi = \phi_1 \wedge \ldots \wedge \phi_d$, then
$(\Lambda^d L)(\phi_1 \wedge \ldots \wedge \phi_d) = L\phi_1 \wedge \ldots \wedge L\phi_d$.

**Proposition 1**

*Let $L \in \mathcal{L}(\mathcal{E}, \mathcal{E})$, then $\Lambda^d L \in \mathcal{L}(\Lambda^d \mathcal{E}, \Lambda^d \mathcal{E})$ and*

$$\omega_d(L) := \|\Lambda^d L\|_{\Lambda^d \mathcal{E}} = \sup_{\phi_1 \wedge \ldots \wedge \phi_d \neq 0} \frac{\|L\phi_1 \wedge \ldots \wedge L\phi_d\|_{\Lambda^d \mathcal{E}}}{\|\phi_1 \wedge \ldots \wedge \phi_d\|_{\Lambda^d \mathcal{E}}} = \sup_{\Pi \subset \mathcal{E}} \frac{vol_d(L\Pi)}{vol_d(\Pi)}$$

# The Liouville formula

$$\frac{d}{dt}\phi(t) = L(t)\phi(t), \quad \phi|_{t=0} = \phi_0,\ t \geqslant 0, \tag{13}$$

for some $L(t) \in L^\infty([0,T];\mathcal{L}(\mathcal{E}))$.

**Proposition 2 (Liouville formula)**

*Let $\phi_1(t),\dots,\phi_d(t)$ be solutions of problem (13), then*

$$\frac{1}{2}\frac{d}{dt}\|\phi_1(t)\wedge\ldots\wedge\phi_d(t)\|^2_{\Lambda^d\mathcal{E}} = Tr(Q(t)L(t)Q(t))\|\phi_1(t)\wedge\ldots\wedge\phi_d(t)\|^2_{\Lambda^d\mathcal{E}},$$

*where $Q(t)$ is the orthoprojector on the $d$-dimensional space spanned by $\phi_1(t),\dots,\phi_d(t)$ and $Tr(\cdot)$ is a trace of the corresponding matrix.*

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## Proposition 2 (Liouville formula)

*Let $\phi_1(t), \ldots, \phi_d(t)$ be solutions of problem (13), then*

$$\frac{1}{2}\frac{d}{dt}\|\phi_1(t) \wedge \ldots \wedge \phi_d(t)\|^2_{\Lambda^d\mathcal{E}} = Tr(Q(t)L(t)Q(t))\|\phi_1(t) \wedge \ldots \wedge \phi_d(t)\|^2_{\Lambda^d\mathcal{E}},$$

*where $Q(t)$ is the orthoprojector on the $d$-dimensional space spanned by $\phi_1(t), \ldots, \phi_d(t)$ and $Tr(\cdot)$ is a trace of the corresponding matrix.*

## Definition 11 (d-trace $Tr_d(L)$)

*Let $L \in \mathcal{L}(\mathcal{E})$, then its $d$-trace $Tr_d(L)$ is defined by*

$$Tr_d(L) = \sup\left\{\sum_{i=1}^{d}(L\psi_i, \psi_i) : \ \psi_i \in \mathcal{E}, \ (\psi_i, \psi_j) = \delta_{ij}\right\}.$$

### Corollary

*Let* $U(t) : \mathcal{E} \to \mathcal{E}$ *be the solution operator of problem* (13), *that is* $\phi(t) = U(t)\phi_0$. *Then* $\omega_d(U(t)) \leqslant exp\{\int_0^t Tr_d(L(s))ds\}$.

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## Proposition 3 ($Tr_d(L)$ representation)

*Let* $L \in \mathcal{L}(\mathcal{E})$ *be positive self-adjoint operator and let*

$$\mu_k(L) = \inf_{F \subset \mathcal{E},\ \dim F = k-1} \ \sup_{\phi \in F^{\perp}, \phi \neq 0} \frac{(L\phi, \phi)}{\|\phi\|^2}.$$

*Then*

$$Tr_d(L) = \sum_{k=1}^{d} \mu_k(L). \tag{14}$$

# The Liouville formula with time dependent metric

*J. - M. Ghidaglia, 1988: Cubic Schrodinger equation on* $[0, L]$*.* $\mathcal{E}(t)$ - Hilbert space $\mathcal{E}$ with family of scalar products $(\cdot,\cdot)_{\mathcal{E}(t)}$:

$$c^{-1}\|\phi\|^2_{\mathcal{E}} \leqslant \|\phi\|^2_{\mathcal{E}(t)} \leqslant c\|\phi\|^2_{\mathcal{E}},\ t \in \mathbb{R}.$$

**Proposition 4 (Liouville formula)**

*Let equation* (13) *be well-posed and its solutions satisfy energy-identity of the form*

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2_{\mathcal{E}(t)} = (M(t)\phi(t), \phi(t))_{\mathcal{E}} = (M_{\mathcal{E}(t)}(t)\phi(t), \phi(t))_{\mathcal{E}(t)} \quad (*)$$

*for some operators* $M(t)$ *and* $M_{\mathcal{E}(t)}(t)$ *defined from Riesz Theorem. Then*

$$\frac{1}{2}\frac{d}{dt}\|\phi_1(t) \wedge \ldots \wedge \phi_d(t)\|^2_{\Lambda^d\mathcal{E}(t)} = Tr(Q(t)M_{\mathcal{E}(t)}(t)Q(t))\|\phi_1(t) \wedge \ldots \wedge \phi_d(t)\|^2_{\Lambda^d\mathcal{E}(t)},$$

*where* $Q(t)$ *is orthoprojector on* $\phi_1(t), \ldots, \phi_d(t)$ *in* $\mathcal{E}(t)$*.*

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### Lemma 1

1. $L \in \mathcal{L}(\mathcal{E})$, *then* $c^{-d}\omega_d(L,\mathcal{E}) \leqslant \omega_d(L,\mathcal{E}(t)) \leqslant c^d\omega_d(L,\mathcal{E})$;
2. $M(t) \geqslant 0$, *then* $c^{-1}Tr_d(M(t),\mathcal{E}) \leqslant Tr_d(M_{\mathcal{E}(t)},\mathcal{E}(t)) \leqslant cTr_d(M(t),\mathcal{E})$.

## Theorem 7 (Main abstract result, A. S., S. Zelik)

*Let linear problem* (13) *be well-posed in* $\mathcal{E}$ *and its solutions* $\phi(t) = U(t)\phi_0$ *possess the energy identity* (*). *Assume*

$$(M(t)\phi, \phi)_{\mathcal{E}} \leqslant (C(t)\phi, \phi)_{\mathcal{E}} + (K(t)\phi, \phi)_{\mathcal{E}}, \ \ \phi \in \mathcal{E}$$

*where*

1.$(C(t)\phi, \phi)_{\mathcal{E}} \leqslant -\beta\|\phi\|^2_{\mathcal{E}}, \ \phi \in \mathcal{E}, \qquad \beta > 0$ *– independent of* $t$;

2.$K(t)$ *– positive,* $(K(t)\phi, \phi) \leqslant (K\phi, \phi), \ K \in \mathcal{L}(\mathcal{E})$ *– compact.*

*Then*

1. $\omega_d(U(t), \mathcal{E}) \leqslant e^{d \ln c + (c\, C_K - \frac{\beta d}{2c})t}$, *where* $C_K$ *depends only on* $K$;
2. *If* $d$ *is such that* $C_K - \beta d/2 < 0$, *then*

$$\omega_d(U(t), \mathcal{E}) \leqslant 1/2, \text{ as long as } t \geqslant t_0.$$

## Proof.

$$\omega_d(U(t), \mathcal{E}) \leqslant c^d \omega_d(U(t), \mathcal{E}(t)) \leqslant$$
$$c^d exp\{\int_0^t Tr_d(C_{\mathcal{E}(s)}(s), \mathcal{E}(s)) + Tr_d(K_{\mathcal{E}(s)}(s), \mathcal{E}(s))ds\} \leqslant$$
$$c^d exp\left\{\int_0^t c^{-1} Tr_d(C(s), \mathcal{E}) + cTr_d(K(s), \mathcal{E})ds\right\} \leqslant$$
$$c^d exp\left\{t\left(-c^{-1}\beta\, d + cTr_d(K, \mathcal{E})\right)\right\} \leqslant$$
$$c^d exp\left\{t\left(-c^{-1}\beta\, d + c(C_K + \beta d\, c^{-2}/2)\right)\right\}.$$

□

# Volume contraction theorem

## Definition 12 (Quasidifferential)

*A map* $S : \mathcal{A} \to \mathcal{A}$*, where* $\mathcal{A}$ *is a compact subset of a Banach space* $\mathcal{E}$ *is called uniform quasidifferentiable on* $\mathcal{A}$ *if for any* $\xi \in \mathcal{A}$ *there exists* $S'(\xi) \in \mathcal{L}(\mathcal{E})$*:*
*1.* $\|S(\xi_2) - S(\xi_1) - S'(\xi_1)(\xi_2 - \xi_1)\|_{\mathcal{E}} = o(\|\xi_1 - \xi_2\|_{\mathcal{E}})$*, holds uniformly for all* $\xi_1, \xi_2 \in \mathcal{A}$*;*
*2.* $S'(\xi) \in C(\mathcal{A}, \mathcal{L}(\mathcal{E}))$*.*

## Theorem 8 (Main abstract result)

*Let* $\mathcal{A}$ *be a compact set of a Hilbert space* $\mathcal{E}$*:*
*1.* $S\mathcal{A} = \mathcal{A}$*; 2.* $S$ *is quasidifferentiable on* $\mathcal{A}$*;*
*3.* $S'(\xi)$ *contacts all d-volumes uniformly w. r. t.* $\xi \in \mathcal{A}$*, i. e.*

$$\omega_d(\mathcal{A}, S) := \sup_{\xi \in \mathcal{A}} \omega_d(S'(\xi), \mathcal{E}) < 1.$$

*Then* $\dim_f^{\mathcal{E}} \mathcal{A} \leqslant d$*.*

V. Chepyzhov, A. Ilyin 2004; R. Temam 1988.

# Quasidifferentiability

**Theorem 9**

*Let $f$ be of subquintic growth and satisfies dissipative assumptions. Then solution operator $S(t)$ of the hyperbolic CHO equation is uniformly quasidifferentiable on the attractor $\mathcal{A}$:*

$$\forall \xi_0 \in \mathcal{A}\ S'(t,\xi_0)\hat{\xi} = \xi_w(t),\ \hat{\xi} \in \mathcal{E},$$

*where $w$ solves the equation of variations*

$$\begin{cases} \partial_t^2 w + \partial_t w + \alpha w + \Delta_x(\Delta_x w - f'(u)w) = 0,\ x \in \mathbb{R}^3, \\ \xi_w|_{t=0} = (w_0, w_0') = \hat{\xi}, \end{cases} \qquad \text{(QD)}$$

*where $\xi_u(t) = S(t)\xi_0$ is $S$-solution hyperbolic CHO equation with initial data $\xi_0 \in \mathcal{A}$.*

Obviously, we can rewrite (QD) as $\frac{d}{dt}\xi_w(t) = L(t,\xi_0)\xi_w(t)$, with

$$L(t,\xi_0) := \begin{pmatrix} 0 & 1 \\ -\alpha & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\Delta_x & 0 \end{pmatrix} \begin{pmatrix} \Delta_x - f'(u(t)) & 0 \\ 0 & 0 \end{pmatrix}. \quad \text{(QD')}$$

Standard argument fails:

$$\frac{1}{2}\frac{d}{dt}\left(\|\xi_w\|^2_{\mathcal{E}} + 2\delta(\partial_t w, w)_{\dot{H}^{-1}} + \delta\|w\|^2_{\dot{H}^{-1}}\right) = \\ -\left((1-\delta)\|\partial_t w\|^2_{\dot{H}^{-1}} + \delta\|w(t)\|^2_{\dot{H}^1} + \alpha\delta\|w(t)\|^2_{\dot{H}^{-1}}\right) \\ -(f'(u)w, \partial_t w) - \delta(f'(u)w, w),$$

where $\delta > 0$ is small enough.

Indeed, assume, for simplicity, that

1.$\Omega$ - smooth and bdd; 2. $\mathcal{E} = H^1_0(\Omega) \times H^{-1}(\Omega)$;

3. $f'(u) \equiv C$,then

$(w, \partial_t w) = (K\xi_w, \xi_w)_{\mathcal{E}},$ *with* $K = \begin{pmatrix} 0 & 0 \\ -\Delta_x & 0 \end{pmatrix}$, and $K \in \mathcal{L}(\mathcal{E})$ is not compact!

# Finite fractal dimension of the attractor: proof

IDEA: kill the $(f'(u)w, \partial_t w)$.
Let us introduce the family of equivalent norms on $\mathcal{E}$:

$$\|\xi_w(t)\|^2_{\mathcal{E}(t,\xi_0)} := \|\xi_w\|^2_{\mathcal{E}} + 2\delta(\partial_t w, w)_{\dot{H}^{-1}} + \delta\|w\|^2_{\dot{H}^{-1}} + (f'(u)w, w) + L\|(-\Delta_x + 1)^{-1/2}(\psi_R w)\|^2_{L^2},$$

where

$0 \leqslant \psi_R(x) \leqslant 1,\ \psi_R(x) = 1 \textit{ if } |x| \leqslant R - 1,\ \psi_R(x) = 0 \textit{ if } |x| \geqslant R,$
and $L$, $R$ are large enough parameters to be fixed below.

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and $L$, $R$ are large enough parameters to be fixed below.

We have

$$\frac{1}{2}\frac{d}{dt}\|\xi_w(t)\|^2_{\mathcal{E}(t,\xi_0)} = -\left((1-\delta)\|\partial_t w\|^2_{\dot{H}^{-1}} + \delta\|w(t)\|^2_{\dot{H}^1} + \alpha\delta\|w(t)\|^2_{\dot{H}^{-1}}\right) - \delta(f'(u)w, w) + \frac{1}{2}(f''(u)\partial_t u, w^2) + L((-\Delta_x+1)^{-1}(\psi_R w), \psi_R \partial_t w) := (M(t,\xi_0)\xi_w, \xi_w)_{\mathcal{E}}. \quad (15)$$

## Theorem 10 (Equivalence of the norms)

*There exist constants* $L,\ R,\ \delta > 0$ *such that*

$$c^{-1}\|\xi_w\|^2_{\mathcal{E}} \leqslant \|\xi_w\|^2_{\mathcal{E}(t,\xi_0)} \leqslant c\|\xi_w\|^2_{\mathcal{E}},$$

*where constant* $c$ *does not depend on* $t$ *and* $\xi_0 \in \mathcal{A}$.

## Proof.

Upper bound: trivial;

Lower bound:

$$(f'(u)w, w) =$$
$$(\psi_R f'(u)w, w) + ((1-\psi_R)[f'(u) - f'(0)]w, w) + f'(0)((1-\psi_R)w, w) \geq$$
$$-C(\psi_R w, w) - C\|(1-\psi_R)u\|_{L^\infty}\|w\|^2_{L^2} \geq$$
$$-C\|(-\Delta_x + 1)^{-1/2}(\psi_R w)\|_{L^2}((-\Delta_x+1)w, w)^{1/2} - C\|(1-\psi_R)u\|_{L^\infty}\|w\|^2_{L^2} \geq$$
$$-C_\varepsilon\|(-\Delta_x+1)^{-1/2}(\psi_R w)\|^2_{L^2} - \varepsilon\|\xi_w\|^2_{\mathcal{E}} - C\|(1-\psi_R)u\|_{L^\infty}\|w\|^2_{L^2},$$

where $\varepsilon > 0$ is arbitrary.

$$\|(1-\psi_R)u\|_{L^\infty} \leqslant \|u\|^{\frac{1}{2}}_{L^2(|x|>R-1)}\|u\|^{\frac{1}{2}}_{H^2} \leqslant \varepsilon_1,$$

as long as $R \geqslant R(\varepsilon_1)$ - large enough.

$$(f'(u)w, w) \geqslant -C_\varepsilon\|(-\Delta_x+1)^{-1/2}(\psi_R w)\|^2_{L^2} - \varepsilon\|\xi_w\|^2_{\mathcal{E}},\ R \geqslant R(\varepsilon).$$

□

# Completion of the proof

$(M(t,\xi_0)\xi_w,\xi_w)_{\mathcal{E}} \leqslant -\gamma\|\xi_w\|^2_{\mathcal{E}} + C\|\psi_R w\|^2 := -\gamma\|\xi_w\|^2_{\mathcal{E}} + (K\xi_w,\xi_w)_{\mathcal{E}}$,
for some $\gamma,\ C>0$ that do not depend on $\xi_0$ and $t$.
Indeed,

$$\begin{aligned}|(f''(u)\partial_t u, w^2)| &\le C(|\partial_t u|,\psi_R w^2) + C((1-\psi_R)|\partial_t u|, w^2) \le\\ &\le C\|\partial_t u\|_{L^3}\|\psi_R w\|_{L^2}\|w\|_{L^6} + C\|(1-\psi_R)\partial_t u\|_{L^3}\|w\|_{L^6}\|w\|_{L^2} \le\\ &\le C\|\psi_R w\|_{L^2}\|\xi_w\|_{\mathcal{E}} + C\|(1-\psi_R)\partial_t u\|_{L^3}\|\xi_w\|^2_{\mathcal{E}}. \quad (16)\end{aligned}$$

$$\begin{aligned}\|(1-\psi_R)\partial_t u\|_{L^3} &\le C\|\partial_t u\|_{L^3(|x|>R-1)} \le\\ &\|\partial_t u\|^{\theta}_{L^2(|x|>R-1)}\|\partial_t u\|^{1-\theta}_{L^6} \leqslant \varepsilon, \quad (17)\end{aligned}$$

as long as $R=R(\varepsilon)$ - large enough. Thus

$$|(f''(u)\partial_t u, w^2)| \le C\varepsilon\|\xi_w\|^2_{\mathcal{E}} + C_\varepsilon\|\psi_R w\|^2_{L^2}, \quad (18)$$

where $\varepsilon>0$ is arbitrary small.

# Final Remark

Remark: Strichartz estimates can be effectively used in dissipative case.
One more example: wave damped equation.

$$\partial_t^2 u + \gamma \partial_t u - \Delta_x u + u|u|^q = g(x),\ x \in \Omega \tag{19}$$

$$\xi_u|_{t=0} = (u_0, u_1) \in \mathcal{E} := H_0^1(\Omega) \times L^2(\Omega),\ u|_{\partial\Omega} = 0. \tag{20}$$

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**Theorem 11**

*Let $\Omega = \mathbb{R}^3$, $0 \leqslant q \leqslant 4$, $\gamma \geqslant 0$, then wave equation possesses a unique energy solution such that $u \in L^5([0, T]; L^{10}(\Omega))$-norm is finite for all $T \geqslant 0$.*

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N. Burq, G. Lebau, F. Planchon, 2008: $\Omega \subset \mathbb{R}^3$ - smooth bdd with boundary;
$q = 4$ : A. Savostianov, V. Kalantarov, S. Zelik, *submitted*.

# Thank You!