On a fractional Cahn-Hilliard equation

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Infinite-dimensional dynamics, dissipative systems, and attractors

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The fractional Cahn-Hilliard model

- We consider the following fractional variant of the [Cahn-Hilliard] system:

$$u_t + (-\Delta)^s w = 0 \quad \text{in } (0,T)\times\Omega, \tag{CH1}$$

$$w = (-\Delta)^\sigma u + \beta(u) - \lambda u \quad \text{in } (0,T)\times\Omega. \tag{CH2}$$

- Fractional diffusion (given by $s, \sigma \in (0,1)$) occurs in both equations;
- The function $\beta : \mathbb{R} \to \mathbb{R}$ is assumed (at least) to be smooth and monotone with $\beta(0) = 0$; $\lambda \geq 0$. Later, we will focus on special cases (e.g., power - like β);
- The system is complemented with the Cauchy condition and with the Dirichlet conditions $u \equiv w \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. The domain Ω is assumed to be smooth and bounded.

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Motivations

- Fractional elliptic and parabolic equations are a rapidly expanding research field, both from the point of view of pure mathematics and of applications;
- The fractional Laplacian can be represented (at least for regular functions) as a spatial convolution with a (nonsmooth) kernel. This situation is coherent with the original [Cahn-Hilliard] model, where a nonlocal (convolution) term replaces the Laplacian in (CH2) in order to describe long-range interactions among particles;
- Our model can yield as singular limits other important diffusion equations of fractional type, like the fractional porous medium (or fast diffusion equation), or the fractional Allen-Cahn equation.

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The fractional Laplacian - 1

- For smooth functions it can be introduced as an integral in the sense of principal value:

$$(-\Delta)^r u(x) := C(r) \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{u(x) - u(y)}{|x - y|^{N+2r}} \, \mathrm{d}y,$$

$C(r) > 0$ being a normalization constant. The limit is required in order to exclude the singularity of the kernel around 0.

- Differently from the standard Laplacian, the value of $(-\Delta)^r u$ at the point x depends on the values of u in the whole of $\mathbb{R}^N$.
- In particular, if u is defined only in Ω, then to compute $(-\Delta)^r u$ it is necessary to extend (e.g. by 0) u outside Ω.

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The fractional Laplacian - 2

- The fractional Laplacian can be defined in several other ways, for instance:
  - by Fourier transform methods (the constant $C(r)$ may come from here);
  - as a fractional power of the standard Laplacian (using, e.g., power series expansions);
  - using spectral theory methods.
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- As one considers functions defined on a bounded domain Ω, this is no longer true. Actually, what we are using is usually referred to as restricted fractional Laplacian.

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How to impose Dirichlet conditions

- We ask for the Dirichlet conditions $u \equiv w \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. This is, indeed, a boundary condition of solid type.
- This formulation is necessary because we need to know the value (e.g.) of u on the whole of $\mathbb{R}^N$ in order to compute $(-\Delta)^\sigma u$.
- Correspondingly, $(-\Delta)^\sigma u(x)$ makes sense for all $x \in \mathbb{R}^N$. Nevertheless, we impose the validity of (CH2), i.e., $w = (-\Delta)^\sigma u + \beta(u) - \lambda u$, only for $x \in \Omega$. The same applies to (CH1).
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Variational form of the Dirichlet fractional Laplacian - 1

- Following, e.g., [Servadei & Valdinoci], we can set

$$\mathcal{X}_{r,0} := \Big\{ v \in L^2(\mathbb{R}^N) : \ v = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

$$\iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2r}} \, \mathrm{d}x \, \mathrm{d}y < +\infty \Big\}.$$

- Correspondingly, we can construct the scalar product

$$\frac{C(r)}{2} \iint_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))(z(x) - z(y))}{|x-y|^{N+2r}} \, \mathrm{d}x \, \mathrm{d}y,$$

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The space $\mathcal{X}_{r,0}$

- Fortunately, $\mathcal{X}_{r,0}$ is nothing very exotic:
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  - For $r \in (0, 1/2)$, $\mathcal{X}_{r,0} \sim H^r(\Omega)$,
  - For $r = 1/2$, $\mathcal{X}_{1/2,0} \sim H_{00}^{1/2}(\Omega)$.
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- For $v, z \in \mathcal{X}_{r,0}$, we set

$$\mathcal{A}_r : \mathcal{X}_{r,0} \to \mathcal{X}'_{r,0},$$

$$\langle \mathcal{A}_r v, z \rangle := \frac{C(r)}{2} \iint_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))(z(x) - z(y))}{|x - y|^{N+2r}} \, \mathrm{d}x \, \mathrm{d}y,$$

- If v, z are smoother (e.g., $v, z \in \mathcal{D}(\Omega)$), then one can easily show that the above is the same as

$$\int_{\mathbb{R}^N} ((-\Delta)^r v)(x) z(x) \, \mathrm{d}x,$$

where $(-\Delta)^r$ is the "strong" fractional Laplacian.

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Weak formulation of the problem

- Replacing the “weak” fractional Laplacians, we obtain

$$u_t + \mathcal{A}_s w = 0 \quad \text{in } \mathcal{X}'_{s,0}, \tag{CH1w}$$

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- Note that this problem can be addressed in a standard Hilbert (Gelfand) triple setting. Indeed, for $r = s, \sigma$, one has the continuous and dense embeddings

$$\mathcal{X}_{r,0} \subset L^2(\Omega) \subset \mathcal{X}'_{r,0},$$

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Existence and uniqueness

Theorem (Well-posedness)

There exist unique functions u, w such that

$$u \in H^1(0,T;\mathcal{X}'_{s,0}) \cap C_w([0,T];\mathcal{X}_{\sigma,0}) \cap C([0,T];L^2(\Omega)),$$
$$w \in L^2(0,T;\mathcal{X}_{s,0}),$$
$$\beta(u) \in L^2((0,T)\times\Omega),$$

satisfying (CH1w)-(CH2w) together with

$$u|_{t=0} = u_0, \quad \text{where } u_0 \in \mathcal{X}_{\sigma,0}, \ \ \widehat{\beta}(u_0) \in L^1(\Omega).$$

We also have the energy inequality (equality if $\sigma \geq s$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u\|^2_{\mathcal{X}_{\sigma,0}} + \int_\Omega \left(\widehat{\beta}(u) - \frac{\lambda}{2}u^2\right)\right) + \|w\|^2_{\mathcal{X}_{s,0}} \leq 0.$$

Strategy of proof and difficulties

- As for the standard [Cahn-Hilliard] model, we use time discretization – a priori estimates – compactness methods;
- There are at least two sources of difficulty:
  - The first is the different order of the fractional operators;
  - The second is that, due to the Dirichlet conditions, the fractional operators cannot be iterated. So, there are limited possibility to obtain parabolic regularization effects by bootstrapping. This is different compared to what happens for other types of fractional Laplace operators.

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General statement

- We focus on the case when $\beta(u) = |u|^{p-1}\,\mathrm{sign}\,u$, where $p \in (1,\infty) \setminus 2$. Moreover, we take $\lambda = 1$ for simplicity. Hence, we end up with

$$u_t + \mathcal{A}_s w = 0, \tag{CH1}$$

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Singular limit: $s \searrow 0$

- For $s \searrow 0$ we formally have $\mathcal{A}_s w_s \to w$; hence,

$$\text{“}\, u_{s,t} + \mathcal{A}_s\big(\mathcal{A}_\sigma u_s + |u_s|^{p-1}\,\mathrm{sign}\,u_s - u_s\big) = 0\,\text{”}$$

$$\to \text{“}\, u_t + \mathcal{A}_\sigma u + |u|^{p-1}\,\mathrm{sign}\,u - u = 0\,\text{”} \qquad \text{(frac:AC)}$$

- Namely, we get in the limit the fractional Allen-Cahn equation.
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- This would yield in the limit the relation

$$u_t + \mathcal{A}_s(|u|^{p-1} \operatorname{sign} u) = 0, \qquad \text{(frac:pm)}$$

corresponding, for $p > 2$, to the fractional porous medium, and, for $p \in (1, 2)$, to the fractional fast diffusion equation.

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Singular limits: key points

- For $s \searrow 0$ (convergence to Allen-Cahn), we keep a uniform estimate of $\{u_s\}$ in $L^\infty(0, T; \mathcal{X}_{0,\sigma})$. This allows us to identify the limit of $|u_s|^{p-1} \operatorname{sign} u_s$ by the Aubin-Lions lemma and monotonicity methods.
- For $\sigma \searrow 0$ (convergence to porous medium), the energy estimate only tells us that (uniformly in time and in σ)

$$\frac{1}{2}\|u_\sigma\|^2_{\mathcal{X}_{0,\sigma}} + \frac{1}{p}\|u_\sigma\|^p_{L^p(\Omega)} - \frac{1}{2}\|u_\sigma\|^2_{L^2(\Omega)} \le c.$$

- Two different cases:
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Limit in the case $p > 2$

- Since $\mathcal{A}_\sigma$ goes to the identity, we do not get strong L^2-convergence of (a subsequence of) u_σ to the limit u.
- Hence, to identify the limit of $|u_\sigma|^{p-1} \operatorname{sign} u_\sigma$, we need to use Minty's trick.
- We are faced to compute (neglect time)

$$\limsup_{\sigma \searrow 0} \left(\|u_\sigma\|_{L^2}^2 - \|u_\sigma\|_{\mathcal{X}_{0,\sigma}}^2 \right)$$

$$\leq \limsup_{\sigma \searrow 0} \left(\lambda_1(\sigma)^{-1} - 1 \right) \|u_\sigma\|_{\mathcal{X}_{0,\sigma}}^2$$

thanks to Poincaré's inequality.

- We can prove that the constant goes to 0 with σ.

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Stationary states

- Let us restrict ourselves to the case $p > 2$.
- For fixed $s, \sigma \in (0,1)$ we may consider the stationary problem

$$\mathcal{A}_s w = 0 \quad \Longrightarrow \quad w = 0$$
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General problem

- We would like to consider general assumptions for β:

$$u_t + \mathcal{A}_s w = 0, \tag{CH1}$$
$$w = \mathcal{A}_\sigma u + \beta(u) - \lambda u. \tag{CH2}$$

We need that $\beta - \lambda\,\mathrm{Id}$ satisfies dissipativity conditions.

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  - For analytic nonlinearity β prove that ω-limits contain a single point, by means of a suitable version of the [Simon-Łojasiewicz] inequality: NOT SO EASY

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Similarities and differences

GOOD: Compared to the standard case, now:

- We can still prove the "second energy estimate" (details later on). Namely, we have parabolic regularization effects;
- We can still rely on the monotonicity of β. Namely, the $L^2(\Omega)$-scalar product $(\mathcal{A}_\sigma u, \beta(u))$ is still nonnegative (at least "formally");

BAD: On the other hand,

- We cannot deduce further regularity by iterating the fractional Laplacian ($\mathcal{A}_{r_1} \circ \mathcal{A}_{r_2} \neq \mathcal{A}_{r_1+r_2}$ due to solid boundary conditions);
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Choice of phase space

- Let us consider the operator

$$\mathcal{Z}_\sigma(u) := \mathcal{A}_\sigma u + \beta(u) - \lambda u.$$

  We need to find suitable spaces X, Y such that $\mathcal{Z}_\sigma : X \to Y$ is analytic.

- For $\beta(u) = u^{p-1}$ ($p \in \mathbb{N}$ even), the natural "energy space" is

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- Only for small p / large σ, i.e., $p \le \frac{2N}{N-2\sigma}$, it turns out that $\mathcal{E}_\sigma = \mathcal{X}_{\sigma,0}$ by Sobolev's embedding; then, we can use $\mathcal{X}_{\sigma,0}$. THAT WORKS!

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The Feireisl-Simondon approach

- In all the other cases, in particular for general β (not power-like), we need a different choice for X and Y.
- For the standard Laplacian, Feireisl-Simondon considered

$$\mathcal{Z}_1 = -\Delta u + \beta(u) - \lambda u, \quad \mathcal{Z}_1 : W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \to L^r(\Omega).$$

- Then, for r large enough, $\mathcal{Z}_1$ is analytic from a neighbourhood U_r of ϕ in $W^{2,r}$ into L^r.

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- Feireisl-Simondon's proof was based on two main points:
    1. Choosing r so large that $W^{2,r} \subset C(\overline{\Omega})$ (fundamental when β grows fast).
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- We take, for suitable $r \geq 2$,

$$X_r = \{u \in \mathcal{X}_{\sigma,0} : \mathcal{A}_\sigma u \in L^r(\Omega)\},$$

endowed with the graph norm.

- Following [Ros-Oton & Serra], even in absence of a well-established regularity theory, it is known that

$$X_r \subset C(\overline{\Omega}) \text{ if } r \geq \frac{N}{2\sigma}. \qquad \text{(good:r)}$$

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Are we allowed to take r satisfying (good:r)?

Eventual uniform boundedness of trajectories

- We need to prove that solutions lie in $\mathcal{X}_r$ uniformly for large times and for r satisfying (good:r).
- The best regularity we can obtain comes from the "second energy estimate": indeed,

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- Then, for $r = \frac{2N}{N-2\sigma}$, applying [Ros-Oton & Serra], we have

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Characterization of ω-limit sets

Theorem (ω-limits)

Let β be monotone, dissipative, and analytic (at least in a neighbourhood of the set of stationary states).

Let also $2s + 4\sigma > N$ and let $r = \frac{2N}{N-2s}$. Then, $\exists\, C \geq 0$ such that, at least for sufficiently large t,

$$\|u(t)\|_{C(\overline{\Omega})} \leq c\|u(t)\|_{X_r} \leq C.$$

Moreover, there exists an equilibrium ϕ such that

$$\lim_{t \nearrow \infty} u(t) = \phi \ \text{ strongly in } \ \mathcal{X}_{\sigma,0} \cap C(\overline{\Omega}).$$

Further developments

- The case of singular potentials (like $W(u) = (1+u)\log(1+u) + (1+u)\log(1+u)$, for $u \in (-1,1)$);
- Second order (in time) models, like the hyperbolic relaxation of the (fractional) Allen-Cahn equation;
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