A gradient flow approach to a fractional porous medium equation

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Joint work with S. Lisini (Pavia) and E. Mainini (Genova)

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The equation

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We consider the problem:

$$\begin{cases} \partial_t u = \operatorname{div}(u \nabla v) & \\ (\alpha I - \Delta)^s v = u & (t,x) \in (0,\infty) \times \mathbb{R}^d. \\ u(0,x) = u_0(x) \in L^1_+(\mathbb{R}^d) & \end{cases}$$

$\alpha \geq 0$ constant, $s \in (0,1)$, u_0 probability density.
u represents a density or a concentration.
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u represents a density or a concentration.
v represents the pressure.
The total mass and positivity of u are formally preserved:

$$\forall t > 0, \quad \int_{\mathbb{R}^d} u(t,x)\,\mathsf{d}x = \int_{\mathbb{R}^d} u_0(x)\,\mathsf{d}x = 1, \quad u(t,x) \geq 0, \ \forall (t,x) \in (0,+\infty)\times\mathbb{R}^d.$$

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   if $0 < s < 1$, we can use the representation

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Some properties:

1. Representation of the inverse:

$$u(x) := \int_{\mathbb{R}^d} f(x-y) \frac{1}{|y|^{d-2s}} \mathsf{d}y = \mathcal{K}_s[f] \quad \text{solves} \quad (-\Delta)^s u = f$$

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2. For $s = 0$ $(-\Delta)^s = \mathrm{I}$, For $s = 1$, $(-\Delta)^s = -\Delta$.

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We consider:

$$\begin{cases} \partial_t u = \text{div}\,(u\mathbf{v}) & \text{Mass Balance} \\ \mathbf{v} = \nabla p & \text{Darcy's Law} \\ p = p(u) = u^\gamma, \gamma > 1, & \text{(local) equation of state} \end{cases}$$

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- More than 60 years of results (Mathchinet indexes ≈ 5000 papers dealing with Porous Medium): well posedness, asymptotics, free boundaries..... by Aronson, Barenblatt Benilan, Brezis, Caffarelli, Crandall, Oleinik, Vázquez.....
- Otto's interpretation (2001) of the PME as a gradient flow w.r.t. the Wasserstein distance (see Ambrosio-Gigli-Savaré book).

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The pressure that drives the flow depends on the whole density, like in the (repulsive) chemotaxis.

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- See J. L. Vazquez & school for results on existence, uniqueness, fine properties of solutions such as infinite speed of propagation (no free boundary)
- See Giulio's talk for relations with a (fractional) Cahn-Hilliard equation

A (formal) kinetic derivation

For any $\varepsilon > 0$, consider in $\mathbb{R}^d \times \mathbb{R}^d \times (0, +\infty)$ the Vlasov equation with friction:

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon}\mathrm{div}_v\big((\nabla_x G_\varepsilon - v) f_\varepsilon\big) = 0 \\ (-\Delta)^s G_\varepsilon = \rho_\varepsilon \\ \rho_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon(t, x, v) \mathsf{d}v \\ f_\varepsilon(0, x, v) = f^0(x, v) \end{cases}$$

The rigorous proof is in Jabin (2000) when $(-\Delta)^s G_\varepsilon = \rho_\varepsilon$ is replaced with $K \star G_\varepsilon = \rho_\varepsilon$ with $K \in L^\infty$.

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with

$$\begin{cases} \partial_t \rho + \mathrm{div}\big(\rho \nabla G\big) = 0 \\ (-\Delta)^s G = \rho \\ \rho(0,x) = \rho^0(x) \end{cases}$$

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Our purpose: give a new existence proof based on the gradient flow interpretation.

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Given $\phi : \mathbb{R}^d \to \mathbb{R}$ smooth function, the *gradient flow* equation is:

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u is a *curve of maximal slope.*

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$\Rightarrow$ Hence is the right concept to deal with

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Given a metric space (X, d), a functional $\phi : X \to (-\infty, +\infty]$ and $u_0 \in Dom(\phi)$, i.e. $\phi(u_0) < +\infty$,

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u is a *curve of maximal slope.*

Example: $X = L^2(\mathbb{R}^n)$

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Given $\psi \in L^2(\mathbb{R}^n)$, we consider the functional

$$\mathscr{F}(u) = \int_{\mathbb{R}^n} \frac{1}{2}\,|\nabla u(x)|^2 + \frac{\alpha}{2}\,u(x)^2 + \psi\,u\;\mathsf{d}x\;,$$

if $u \in H^1(\mathbb{R}^n)$ and $\mathscr{F}(u) = +\infty$ otherwise.

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if $u \in H^1(\mathbb{R}^n)$ and $\mathscr{F}(u) = +\infty$ otherwise.
For every $u_0 \in H^1(\mathbb{R}^n)$ the gradient flow of $\mathscr{F}$ exists and satisfies the equation

$$\partial_t u = \Delta u - \alpha u - \psi, \qquad u(0,\cdot) = u_0(\cdot).$$

Moreover the following dissipation identity holds

$$\frac{\mathsf{d}}{\mathsf{d}t}\mathscr{F}(u) = -\int_{\mathbb{R}^n} |\Delta u - \alpha u - \psi|^2 \, \mathsf{d}x \,.$$

Wasserstein distance (Kantorovitch-Rubinstein-Wasserstein)

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Given μ, $\nu \in \mathscr{P}_2(\mathbb{R}^d)$, we define

$$W(\mu, \nu) := \left(\inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\gamma(x, y) : \gamma \in \Gamma(\mu, \nu) \right\} \right)^{\frac{1}{2}},$$

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$(\mathscr{P}_2(\mathbb{R}^d), W)$ is a complete and separable metric space.

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For $\psi : \mathbb{R}^d \to \mathbb{R}$ we define the functional $\mathscr{G} : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ by

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If ψ is convex or, for instance, $\psi \in C_c^2(\mathbb{R}^n)$ then for every $u_0 \in D(\mathscr{G})$ the scheme associated to $\mathscr{G}$ converges to a unique solution of the Fokker-Planck equation

$$\partial_t u = \Delta u + \operatorname{div}(u \nabla \psi), \qquad u(0, \cdot) = u_0(\cdot).$$

The dissipation identity holds

$$\frac{\mathsf{d}}{\mathsf{d}t}\mathscr{G}(u) = -\int_{\mathbb{R}^n} \left| \frac{\nabla u + u \nabla \psi}{u} \right|^2 u \, \mathsf{d}x \, .$$

(Jordan-Kinderlehrer-Otto 1998, Ambrosio-Gigli-Savaré 2005)

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Let $\mathscr{G} : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ be the integral functional

$$\mathscr{G}(u) = \int_{\mathbb{R}^d} F(u) + \psi u \,\mathsf{d}x + \frac{1}{2}\int_{\mathbb{R}^d} K * u u \,\mathsf{d}x \ ,$$

where K is a function of the form $K(x) = \tilde{K}(|x|)$, and F is a convex function (superlinear at ∞).

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$$\partial_t u = \Delta L_F(u) + \operatorname{div}(u \nabla \psi) + \operatorname{div}(u \nabla (K * u)), \qquad u(0, \cdot) = u_0(\cdot),$$

where $L_F(u) = uF'(u) - F(u)$.
The dissipation identity holds

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(Carrillo-McCann-Villani(2003, 2006), Ambrosio-Gigli-Savaré 2005)

Fractional porous medium equation as gradient flow of an interaction functional

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Using the Fourier transform, the second equation can be written as

$$(\alpha + |\xi|^2)^s \hat{v}(\xi) = \hat{u}(\xi)$$

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$$v = K * u, \qquad \text{where} \qquad \hat{K}(\xi) = (\alpha + |\xi|^2)^{-s}.$$

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The equation can be rewritten as

$$\partial_t u = \operatorname{div}(u \nabla (K * u)).$$

This equation is in the form of gradient flow of the interaction potential

$$\mathscr{F}(u) = \frac{1}{2} \int_{\mathbb{R}^d} K * u \, u \, \mathsf{d}x$$

in the space $\mathscr{P}_2(\mathbb{R}^d)$ with respect to the Wasserstein distance.

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in the space $\mathscr{P}_2(\mathbb{R}^d)$ with respect to the Wasserstein distance.
In the case $\alpha = 0$ one has $K(x) = C|x|^{2s-d}$. Since the Kernel is singular and the basic theory of gradient flows of interaction potentials does not apply.

Gradient flow interpretation of the equation

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The formal computation of the energy dissipation yields

$$\frac{\mathsf{d}}{\mathsf{d}t}\mathscr{F}(u) = -\int_{\mathbb{R}^d} \left|\nabla(K * u)\right|^2 u \,\mathsf{d}x.$$

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When the functional $\mathscr{F}$ is convex along geodesics of the Wasserstein space, the energy dissipation identity is satisfied by the limit curve of the minimising movement scheme. In our case $\mathscr{F}$ is not convex and we only hope to obtain an inequality.

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The functional can be rewritten in Fourier variables as

$$\mathscr{F}(u) = \frac{1}{2}\int_{\mathbb{R}^d} K * u\, u \,\mathsf{d}x = \frac{1}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} (\alpha + |\xi|^2)^{-s} |\hat{u}(\xi)|^2 \,\mathsf{d}\xi,$$

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Proposition (Properties of the functional)

The functional $\mathscr{F}$ satisfies:

- *if $\alpha > 0$ then $D(\mathscr{F}) = H^{-s}(\mathbb{R}^d) \cap \mathscr{P}_2(\mathbb{R}^d)$;*
- *if $\alpha = 0$ then $D(\mathscr{F}) = \dot{H}^{-s}(\mathbb{R}^d) \cap \mathscr{P}_2(\mathbb{R}^d)$;*
- *$\mathscr{F}(u) \geq 0$;*
- *$\mathscr{F}$ is sequentially lower semi continuous with respect to the narrow convergence.*

Well posedness of the scheme

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The approximation scheme is

$$u_\tau^k \text{ minimises in } \mathscr{P}_2(\mathbb{R}^d) \text{ the functional} \quad u \mapsto \frac{1}{2\tau} W^2(u, u_\tau^{k-1}) + \mathscr{F}(u) \ ,$$

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Proposition

For every $\tau > 0$ and every $u_0 \in D(\mathscr{F})$ there exists a unique sequence of minimisers $\{u_\tau^k\} \subset D(\mathscr{F})$ satisfying $u_\tau^0 = u_0$.

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The proof is based on standard lower semicontinuity and compactness of the sublevels.

The basic energy estimate

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In the general scheme

$$\frac{W^2(u_\tau^k, u_\tau^{k-1})}{2\tau} + \mathscr{F}(u_\tau^k) \leq \frac{W^2(u, u_\tau^{k-1})}{2\tau} + \mathscr{F}(u),$$

choosing $u := u_\tau^{k-1}$ we obtain

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Summing from $k = 1$ to $k = N$ we obtain

$$\mathscr{F}(u_\tau^N) + \frac{\tau}{2}\sum_{k=1}^{N}\frac{W^2(u_\tau^k, u_\tau^{k-1})}{\tau^2} \le \mathscr{F}(u_0), \qquad \forall N \in \mathbb{N}.$$

Compactness result

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Denoting by u_τ the piecewise constant approximate solution

$$u_\tau(t) = u_\tau^n, \qquad \text{if } t \in ((n-1)\tau, n\tau],$$

it holds (recall $\int_{\mathbb{R}^d} |x|^2 v(x) \mathsf{d}x = W^2(v, \delta_0)$)

$$\int_{\mathbb{R}^d} |x|^2 u_\tau(t)(x)\, \mathsf{d}x \leq 2T\mathscr{F}(u_0) + 2\int_{\mathbb{R}^d} |x|^2 u_0(x)\, \mathsf{d}x, \qquad \forall t \in [0, T].$$

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For the bound on the second moments stated above we obtain that:

The set $\{u_\tau(t) : (t, \tau) \in [0, T] \times (0, 1)\}$ is contained in a narrowly compact subset of $\mathscr{P}_2(\mathbb{R}^d)$ (and a weakly compact subset of $H^{-s}(\mathbb{R}^d)$).

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Using the basic estimate we obtain also the following equicontinuity result:

There exists a constant C such that

$$\limsup_{\tau\to 0} W(u_\tau(t_1), u_\tau(t_2)) \le C\sqrt{|t_1 - t_2|}\,, \qquad \forall\,(t_1,t_2) \in [0,T]^2.$$

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A suitable version of Ascoli-Arzelà theorem shows the existence of a limit curve (candidate to be a solution).

Existence of a limit curve

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Theorem (Compactness and convergence of the scheme)

For every vanishing sequence τ_n *there exists a subsequence* τ_{n_k} *and a curve* $u \in AC^2([0,+\infty);(\mathscr{P}_2(\mathbb{R}^d),W))$ *such that*

$$u_{\tau_{n_k}}(t) \to u(t) \quad \text{narrowly as } k \to \infty, \text{ for any } t \in [0,+\infty).$$

Moreover

$$\sup_{t\in[0,T]} \left\{ \int_{\mathbb{R}^d} |x|^2 u(t)(x)\,\mathsf{d}x + \mathscr{F}(u(t)) \right\} < +\infty.$$

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Main idea: first variation of the minimum problem along gradient flows generated by convex functionals.

Flow interchange at a discrete level

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Let $\mathscr{H} : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ be the logarithmic entropy

$$\mathscr{H}(u) = \int_{\mathbb{R}^d} u \log u \, \mathsf{d}x.$$

$\mathscr{H}$ is a l.s.c. functional that admits a continuous semigroup $(S_t)_{t\geq 0}$ satisfying the following *Evolution Variational Inequality* (EVI)

$$\frac{1}{2}\frac{W^2(S_t(u),\bar{u}) - W^2(u,\bar{u})}{t} + \mathscr{H}(S_t(u)) \leq \mathscr{H}(\bar{u}), \qquad \forall\, u, \bar{u} \in D(\mathscr{H}),\ t > 0.$$

and $S_t(u)$ solves the heat equation with initial datum u.

Flow interchange at a discrete level

By the minimising scheme, for any $u \in D(\mathscr{H})$

$$\frac{1}{2\tau} W^2(u_\tau^n, u_\tau^{n-1}) + \mathscr{F}(u_\tau^n) \le \frac{1}{2\tau} d^2(u, u_\tau^{n-1}) + \mathscr{F}(u) \, .$$

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As $\mathscr{H}$ satisfies the EVI we have

$$\frac{\mathscr{F}(u_\tau^n) - \mathscr{F}(S_t(u_\tau^n))}{t} \leq \frac{\mathscr{H}(u_\tau^{n-1}) - \mathscr{H}(S_t(u_\tau^n))}{\tau} \ .$$

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Since $\mathscr{H}$ is lower semi continuous, passing to the limit as $t \to 0$ we obtain

$$\limsup_{t \to 0} \frac{\mathscr{F}(u_\tau^n) - \mathscr{F}(S_t(u_\tau^n))}{t} \leq \frac{\mathscr{H}(u_\tau^{n-1}) - \mathscr{H}(u_\tau^n)}{\tau} \,.$$

Improved regularity

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Theorem (Improved regularity of minimizers and decay of the entropy)

Let $u_0 \in D(\mathscr{F}_s) \cap D(\mathscr{H})$ and u_τ^k the solutions of the minimum problem. Then

$$\mathscr{H}(u_\tau^k) \leq \mathscr{H}(u_\tau^{k-1}).$$

Moreover $u_\tau^k \in H^{1-s}(\mathbb{R}^d)$ and there exists a constant C independent of k, τ and s such that

$$\|u_\tau^k\|^2_{H^{1-s}(\mathbb{R}^d)} \leq C + C\frac{\mathscr{H}(u_\tau^{k-1}) - \mathscr{H}(u_\tau^k)}{\tau}. \tag{1}$$

Euler-Lagrange equation

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The first variation of the minimum problem along the perturbation

$$u_\delta := (I + \delta\,\eta)_\# u_\tau^k\ ,$$

for $\eta \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$, using the regularity of the minimizers, yields

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Proposition (Euler-Lagrange equation)

Let $u_0 \in D(\mathscr{F}_s) \cap D(\mathscr{H})$. Let u_τ^k be the minimizer of the scheme and $v_\tau^k := K * u_\tau^k$. Then there holds

$$\int_{\mathbb{R}^d} \nabla v_\tau^k \cdot \eta\, u_\tau^k \,\mathrm{d}x = \frac{1}{\tau}\int_{\mathbb{R}^d} (\mathbf{T}_\tau^k - \mathbf{I})\cdot \eta\, u_\tau^k \,\mathrm{d}x, \qquad \forall\, \eta \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d), \tag{2}$$

where $\mathbf{T}_\tau^k$ denotes the optimal transport map between u_τ^k and u_τ^{k-1} and $\mathbf{I}$ is the identity map on $\mathbb{R}^d$.
Moreover it holds

$$\int_{\mathbb{R}^d} |\nabla v_\tau^k|^2 u_\tau^k \,\mathsf{d}x = \frac{1}{\tau^2} W^2(u_\tau^k, u_\tau^{k-1}).$$

Sketch of the proof of Euler-Lagrange equation

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We use the following notation: $u := u_\tau^k$ and $u_\delta := (\Phi_\delta)_\# u$.
By the minimum problem we have for $\delta > 0$

$$0 \le \frac{1}{\delta}\left(\mathscr{F}(u_\delta) - \mathscr{F}(u)\right) + \frac{1}{\delta}\left(\frac{1}{2\tau}W^2(u_\delta, u_\tau^{k-1}) - \frac{1}{2\tau}W^2(u, u_\tau^{k-1})\right)$$

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It is a standard computation

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Finally

$$0 \leq \int_{\mathbb{R}^d} \nabla v \cdot \eta u \mathrm{d}x - \frac{1}{\tau}\int_{\mathbb{R}^d} (T_\tau^k - I)\cdot \eta u \mathrm{d}x.$$

The above inequality, being valid also for $-\eta$ instead of η, is indeed an equality.

Space-time estimates

Space-time estimates

From the basic energy estimate

$$\mathscr{F}(u_\tau^N) + \frac{\tau}{2}\sum_{n=1}^{N} \frac{W^2(u_\tau^n, u_\tau^{n-1})}{\tau^2} \le \mathscr{F}(u_0)$$

and the consequence of the Euler-Lagrange equation

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Lemma (Time integrated estimates)

Let $u_0 \in D(\mathscr{F}) \cap D(\mathscr{H})$ and $u_\tau(t) := u_\tau^{\lceil t/\tau \rceil}$ the discrete piecewise constant approximate solution. Then we have

$$\int_0^T \int_{\mathbb{R}^d} |\nabla v_\tau|^2 u_\tau \,\mathsf{d}x\,\mathsf{d}t \le 2\mathscr{F}(u_0).$$

Moreover it holds

$$\int_0^T \|u_\tau(t)\|^2_{H^{1-s}(\mathbb{R}^d)} \,\mathsf{d}t \le C(1 + \mathscr{H}(u_0) + T\mathscr{F}(u_0) + \int_{\mathbb{R}^d} |x|^2 u_0(x)\,\mathrm{d}x),$$

where C is a constant depending only on α and d.

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Theorem (The limit solves the equation)

Given $u_0 \in D(\mathscr{F}_s) \cap D(\mathscr{H})$, the limit curve u satisfies the following properties:

(i) *For every vanishing sequence τ_n there exists a subsequence τ_{n_k} and a curve $u \in AC^2([0,+\infty);(\mathscr{P}_2(\mathbb{R}^d),W))$ such that*

$$u_{\tau_{n_k}}(t) \to u(t) \quad \textit{narrowly as } k \to \infty, \textit{ for any } t \in [0,+\infty).$$

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(ii) $u \in L^2((0,T); H^{1-s}(\mathbb{R}^d))$ *and*

$$u_{\tau_{n_k}} \to u \quad \text{strongly in } L^2((0,T); L^2_{loc}(\mathbb{R}^d)) \qquad \text{as } k \to \infty, \forall T > 0.$$

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$$u_{\tau_{n_k}} \to u \quad \text{strongly in } L^2((0,T); L^2_{loc}(\mathbb{R}^d)) \qquad \text{as } k \to \infty, \forall\, T > 0.$$

(iii) *Denoting by $v(t) = K * u(t)$ for $t > 0$, we have $\nabla v \in L^2((0,T); L^2(\mathbb{R}^d))$ for every $T > 0$ and the equation is satisfied in the following form:*

$$\int_0^{+\infty} \int_{\mathbb{R}^d} \partial_t \varphi u - \nabla\varphi \cdot \nabla v u \, \mathrm{d}x \, \mathrm{d}t = 0, \quad \text{for all } \varphi \in C_c^\infty((0,+\infty) \times \mathbb{R}^d).$$

Energy dissipation inequality

Energy dissipation inequality

As a consequence of the Euler-Lagrange equation and the basic energy estimate it holds

$$\mathscr{F}(u_\tau^N) + \frac{1}{2}\int_0^{N\tau}\int_{\mathbb{R}^d} |\nabla v_\tau|^2 u_\tau \, \mathsf{d}x \, \mathsf{d}t \le \mathscr{F}(u_0).$$

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Passing to the limit as $\tau \to 0$ and using a lower semi continuity argument we obtain

$$\mathscr{F}(u(T)) + \frac{1}{2}\int_0^{T}\int_{\mathbb{R}^d} |\nabla v|^2 u \, \mathsf{d}x \, \mathsf{d}t \leq \mathscr{F}(u_0).$$

This is a non sharp energy inequality: there is 1/2 in front of the integral.

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$$\tilde{u}_\tau(t) = \mathrm{Argmin}_{u \in \mathscr{P}_2(\mathbb{R}^d)} \left\{ \frac{1}{2(t-(k-1)\tau)} W^2(u, u_\tau^{k-1}) + \mathscr{F}(u) \right\}, \qquad t \in ((k-1)\tau,$$

For the functional $\mathscr{F}$ this interpolant is uniquely defined and $\tilde{u}_\tau(k\tau) = u_\tau^k$ for any $k \in \mathbb{N}$.

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The one step energy identity can be proved:

$$\frac{1}{2}\frac{W^2(u_\tau^k, u_\tau^{k-1})}{\tau} + \frac{1}{2}\int_{(k-1)\tau}^{k\tau} \frac{W^2(\tilde{u}_\tau(t), u_\tau^{k-1})}{(t-(k-1)\tau)^2}\,\mathsf{d}t + \mathscr{F}_{(}u_\tau^k) = \mathscr{F}_{(}u_\tau^{k-1}).$$

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For k such that $t \in ((k-1)\tau, k\tau]$ we have

$$\int_{\mathbb{R}^d} |\nabla \tilde{v}_\tau(t)|^2 \tilde{u}_\tau(t)\,\mathrm{d}x = \frac{1}{(t-(k-1)\tau)^2} W^2(\tilde{u}_\tau(t), u_\tau^{k-1}).$$

and we obtain

$$\frac{1}{2}\int_0^{N\tau}\int_{\mathbb{R}^d} |\nabla v_\tau|^2\, u_\tau\, \mathsf{d}x\, \mathsf{d}t + \frac{1}{2}\int_0^{N\tau}\int_{\mathbb{R}^d} |\nabla \tilde{v}_\tau|^2\, \tilde{u}_\tau\, \mathsf{d}x\, \mathsf{d}t + \mathscr{F}(u_\tau(N\tau)) = \mathscr{F}(u_0).$$

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Since

$$d^2(\tilde{u}_h(t), u_h(t)) \le C(T)h, \qquad \forall t \in [0,T],$$

passing to the limit as $\tau \to 0$ we obtain the energy inequality for the limit solution.

Energy dissipation inequality

Energy dissipation inequality

Theorem (Dissipation inequality)

The solution u furthermore satisfies: for all $T > 0$

$$\mathscr{F}(u(T)) + \int_0^T \int_{\mathbb{R}^d} |\nabla v(t)|^2 u(t) \, \mathrm{d}x \, \mathrm{d}t \leq \mathscr{F}(u_0), \qquad \forall \, T > 0.$$

Decay of the entropy and of the L^p norm

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Using the flow interchange lemma we obtain the following

Theorem (Decay)

The solution u furthermore satisfies:

$$\mathscr{H}(u(t)) \leq \mathscr{H}(u_0), \qquad \forall t > 0.$$

If $u_0 \in L^p(\mathbb{R}^d)$ for $p \in [1, +\infty]$, then

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \leq \|u_0\|_{L^p(\mathbb{R}^d)}, \qquad \forall t > 0.$$

Limit for $s \to 0$

Limit for $s \to 0$

Theorem

Let $u_0 \in L^2(\mathbb{R}^d)$. Denoting by u^s the solutions of the equation with parameter s and by u the unique solution of the porous medium equation

$$\partial_t u - \frac{1}{2}\Delta(u^2) = 0 \quad \text{in} \quad \mathbb{R}^d \times (0, +\infty),$$

with initial datum u_0 satisfying

$$\frac{1}{2}\|u(T)\|^2_{L^2(\mathbb{R}^d)} + \int_0^T \int_{\mathbb{R}^d} |\nabla u(t)|^2 u(t)\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)}, \qquad \forall\, T > 0,$$

we have

$$u^s(t) \to u(t) \quad \text{narrowly as } s \to 0 \text{ for every } t \geq 0,$$

$$u^s \to u \quad \text{strongly in } L^2((0,T); L^2_{loc}(\mathbb{R}^d)) \text{ as } s \to 0 \text{ for every } T > 0.$$

Open Problems

- Uniqueness
  - For $s = 1$ uniqueness holds for $u_0 \in L^\infty$ (Ambrosio, Serfaty)
  - For $s = 0$ uniqueness holds (standar PM)
- Relations with Caffarelli & Vazquez solution
  - From L^1 to $L \log L$ estimates
  - Regularity