Controllability implies ergodicity

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Two examples

Example

Let us consider the following ODE on a compact Riemannian manifold X:

$$\dot{u} = V_0(u) + \sum_{j=1}^{m} \eta^j(t) V_j(u), \quad u(t) \in X. \tag{1}$$

Here $V_0, \dots, V_m$ are smooth vector fields on X and $\eta^j(t)$ are scalar random processes. Under mild regularity assumptions on η^j, the Cauchy problem for (1) is well posed, and the solutions are random processes with range in X.

Two examples

Example

Let us consider the Navier–Stokes system in a bounded domain $D \subset \mathbb{R}^2$:

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = h(x), \quad \operatorname{div} u = 0, \quad x \in D, \qquad (2)$$
$$u\big|_{\partial D} = \eta.$$

Here η is an $\mathbb{R}^2$-valued sufficiently regular random field on $\mathbb{R}_+ \times \partial D$. Assuming that η is bounded, on can prove that the random flow generated by (2) possesses a compact invariant absorbing set $X \subset L^2(D, \mathbb{R}^2)$. For any initial state $u_0 \in X$, the corresponding solution of (2) is a random process in X.

Problem: Asymptotic behaviour of solutions as $t \to \infty$.

Outline

Total variation distance

Let X be a compact metric space, $\mathcal{P}(X)$ the space of probability measures on X, and $\mathcal{D}(\xi)$ the law of a random variable ξ.

Definition

For any $\mu_1, \mu_2 \in \mathcal{P}(X)$, define the total variation distance

$$\|\mu_1 - \mu_2\|_{\mathrm{var}} = \sup_{\Gamma \in \mathcal{B}(X)} |\mu_1(\Gamma) - \mu_2(\Gamma)|,$$

where $\mathcal{B}(X)$ is the set of Borel subsets of X.

Lemma

For any $\mu_1, \mu_2 \in \mathcal{P}(X)$*, there are* $\mu, \mu_1', \mu_2' \in \mathcal{P}(X)$ *such that*

$$\mu_1 = (1-d)\mu + d\mu_1', \quad \mu_2 = (1-d)\mu + d\mu_2' \tag{3}$$

where $d = \|\mu_1 - \mu_2\|_{\mathrm{var}}$.

Maximal coupling

Corollary

For any $\mu_1, \mu_2 \in \mathcal{P}(X)$, there is a pair of X-valued random variables (ξ_1, ξ_2) (called maximal coupling) such that

$$\mathcal{D}(\xi_1) = \mu_1, \quad \mathcal{D}(\xi_2) = \mu_2, \tag{4}$$

$$\mathbb{P}\{\xi_1 \neq \xi_2\} = \|\mu_1 - \mu_2\|_{\mathrm{var}}. \tag{5}$$

Moreover, for any pair (ξ_1, ξ_2) satisfying (4) we have

$$\mathbb{P}\{\xi_1 \neq \xi_2\} \geq \|\mu_1 - \mu_2\|_{\mathrm{var}}. \tag{6}$$

Remark

The corollary shows that the total variation distance between two measures μ_1 and μ_2 is the minimal probability with which two random variables with those laws must be different.

Discrete-time random dynamical systems

Let X be a compact metric space, E separable Hilbert space, and $S : X \times E \to X$ a continuous mapping. Consider the RDS

$$u_k = S(u_{k-1}, \eta_k), \quad k \geq 1, \tag{7}$$

where $\{\eta_k\}$ is a sequence of i.i.d. random variables in E.

Example

Consider an ODE on a compact Riemannian manifold X:

$$\dot{u} = V_0(u) + \sum_{j=1}^{m} \eta^j(t) V_j(u). \tag{8}$$

Setting $\eta = (\eta^1, \ldots, \eta^m)$ and $\eta_k = \eta|_{[k-1,k]}$, we write

$$u(k) = S\big(u(k-1), \eta_k\big), \quad S : \big(u_0, \eta\big|_{[0,1]}\big) \mapsto u(1).$$

Main result

Define the transition function

$$P_k(v,\Gamma) = \mathbb{P}_v\{u_k \in \Gamma\}, \quad v \in X, \quad \Gamma \in \mathcal{B}(X).$$

Theorem

Suppose $\exists\, \hat{u} \in X \;\exists\, r > 0$ *with the following properties:*

Recurrence: $\exists\, m \geq 1 \;\exists\, p > 0$ *such that*

$$P_m(u, B_X(\hat{u}, r)) \geq p \quad \forall u \in X. \tag{9}$$

Coupling: $\exists\, \varepsilon > 0$ *such that*

$$\|P_1(u,\cdot) - P_1(u',\cdot)\|_{\mathrm{var}} \leq 1 - \varepsilon \quad \forall u, u' \in B_X(\hat{u}, r). \tag{10}$$

Then $\exists\, C, \gamma > 0$ *and a unique measure* $\mu \in \mathcal{P}(X)$ *such that*

$$\|\mathcal{D}_\lambda(u_k) - \mu\|_{\mathrm{var}} \leq C\, e^{-\gamma k} \quad \forall k \geq 0 \quad \forall \lambda \in \mathcal{P}(X). \tag{11}$$

Differential equations with random coefficients

Consider an ODE on a compact Riemannian manifold X:

$$\dot{u} = V_0(u) + \sum_{j=1}^{m} \eta^j(t) V_j(u). \tag{12}$$

We assume that

$$\eta(t) = (\eta^1(t), \ldots, \eta^m(t)) = \sum_{k=1}^{\infty} I_{[k-1,k)}(t)\eta_k(t-k+1), \tag{13}$$

where $\{\eta_k\}$ are i.i.d. random variables in $E := L^2(0,1;\mathbb{R}^m)$.

The Cauchy problem for (12) is well posed, and all solutions exist globally in time for any initial state $v \in X$.

Main result for ODE's

Theorem

Assume that the following two hypotheses are satisfied:

Hörmander condition: $\mathrm{Lie}_u(V_1, \dots, V_m) = T_u X$ *for any* $u \in X$.
Non-degeneracy: There is an ONB $\{e_i\}$ *in E such that*

$$\eta_k(t) = \sum_{i=1}^{\infty} \xi_{ik} e_i(t) \qquad (14)$$

where $\{\xi_{ik}\}$ *are independent random variables satisfying*

$$\mathcal{D}(\xi_{ik}) = \rho_i(r)\mathrm{d}r; \quad \rho_i \in C(\mathbb{R}), \quad \rho_i > 0; \quad \sum_{i=1}^{\infty} \mathbb{E}\, \xi_{ik}^2 < \infty.$$

Then $\exists$ *a unique* $\mu \in \mathcal{P}(X)$ *and* $\exists\, C, \gamma > 0$ *such that*

$$\|\mathcal{D}_\lambda(u(k)) - \mu\|_{\mathrm{var}} \leq C\, e^{-\gamma k} \quad \forall k \geq 0 \quad \forall \lambda \in \mathcal{P}(X). \qquad (15)$$

Kantorovich–Wasserstein and total variation metrics

Define the space of uniformly continuous bounded functions

$$L_b(X) = \{f \in C_b(X) : |f(u) - f(v)| \leq C\, d_X(u,v)\ \forall\, u, v \in X\}.$$

Denote by $\|\cdot\|_L$ the natural norm on $L_b(X)$ and define

$$\|\mu_1 - \mu_2\|_L^* = \sup_{\|f\|_L \leq 1} |(f,\mu_1) - (f,\mu_2)|, \quad (f,\mu) = \int_X f \mathrm{d}\mu.$$

It can be proved that

$$\|\mu_1 - \mu_2\|_{\mathrm{var}} = \frac{1}{2} \sup_{\|f\|_\infty \leq 1} |(f,\mu_1) - (f,\mu_2)|,$$

so that the KW metric is (much) weaker than the TV metric.

Example

If $u, v \in X$ and $u \neq v$, then

$$\|\delta_u - \delta_v\|_{\mathrm{var}} = 1, \quad \|\delta_u - \delta_v\|_L^* = d_X(u,v).$$

General criterion

Consider the random dynamical system

$$u_k = S(u_{k-1}, \eta_k), \quad u_k \in X, \quad \eta_k \in E.$$

Theorem

Suppose the following two conditions hold for some $\hat{u} \in X$:

Recurrence: For any $\delta > 0$ there are $p > 0$, $m \geq 1$ such that

$$P_m(u, B(\hat{u}, \delta)) \geq p \quad \text{for any } u \in X. \tag{16}$$

Stability: $\exists$ function $\varepsilon(r) > 0$ going to zero as $r \to 0^+$ such that

$$\sup_{k \geq 0} \| P_k(u, \cdot) - P_k(u', \cdot) \|_L^* \leq \varepsilon(r) \quad \forall\, u, u' \in B_X(\hat{u}, r). \tag{17}$$

Then there is a unique $\mu \in \mathcal{P}(X)$ such that

$$\sup_{\lambda \in \mathcal{P}(X)} \| \mathcal{D}_\lambda(u_k) - \mu \|_L^* \to 0 \quad \text{as } k \to \infty. \tag{18}$$

Sufficient condition for stability

Proposition

Suppose the following condition holds for some C and $q < 1$:

Squeezing: For any $u, u' \in X$ there are X-valued random variables v, v' such that

$$\mathbb{P}\{d_X(v, v') \le q\, d_X(u, u')\} \ge 1 - C\, d_X(u, u'), \tag{19}$$

$$\mathcal{D}(v) = P_1(u, \cdot), \quad \mathcal{D}(v') = P_1(u', \cdot). \tag{20}$$

Then the stability holds for any $\hat{u} \in X$.

Remark

It can be proved that if the recurrence and squeezing conditions are satisfied, then we have the exponential convergence

$$\sup_{\lambda \in \mathcal{P}(X)} \|\mathcal{D}_\lambda(u_k) - \mu\|_L^* \le C\, e^{-\gamma k} \quad \forall k \ge 0.$$

Idea of the proof of the proposition

Let us denote by $\mathcal{R}(u,u')$ and $\mathcal{R}'(u,u')$ the random variables v and v' satisfying (19) and (20). Define a sequence (v_k, v_k') by

$$
\begin{aligned}
v_0 &= u, & v_0' &= u',\\
v_k &= \mathcal{R}(v_{k-1}, v_{k-1}'), & v_k' &= \mathcal{R}'(v_{k-1}, v_{k-1}').
\end{aligned}
$$

Then $\mathcal{D}(v_k) = P_k(u,\cdot)$, $\mathcal{D}(v_k') = P_k(u',\cdot)$ for $k \ge 0$. Moreover,

$$
\mathbb{P}\{d_X(v_k, v_k') \le 2^{-k} d_X(u,u) \text{ for all } k \ge 0\} \ge 1 - C\,\|u-u'\|. \quad (21)
$$

For any bounded 1-Lipschitz function $f : X \to \mathbb{R}$, we have

$$
\begin{aligned}
\big|(f, P_k(u,\cdot)) - (f, P_k(u',\cdot))\big| &= \big|\mathbb{E}\big(f(v_k) - f(v_k')\big)\big|\\
&= \big|\mathbb{E}\big(f(v_k) - f(v_k')\big)(I_G + I_{G^c})\big|\\
&\le 2^{-k} d_X(u,u') + C\, d_X(u,u').
\end{aligned}
$$

Navier–Stokes system with a boundary noise

Let $D \subset \mathbb{R}^2$ and $D_1 = (0,1) \times D$. Consider the problem

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = h(t,x), \quad \operatorname{div} u = 0, \tag{22}$$

$$u\big|_{\partial D} = \eta(t,x), \tag{23}$$

$$u(0,x) = u_0(x). \tag{24}$$

Here h is a 1-periodic function belonging to $H^1_{\mathrm{loc}}(\mathbb{R}_+ \times D)$ and $\eta(t,x)$ is a space-time localised noise of the form

$$\eta(t,x) = \sum_{k=1}^{\infty} I_{[k-1,k)}(t) \eta(t-k+1, x), \tag{25}$$

where $\{\eta_k\}$ is a sequence i.i.d. random variables in the space

$$E = \Big\{ g \in H^{5/2}((0,1) \times \partial D) : \int_{\partial D} \langle g(t), \boldsymbol{n} \rangle \mathrm{d}\sigma_x \equiv 0, \operatorname{supp} g \subset Q \Big\},$$

where $Q \Subset (0,1) \times \partial D$.

Navier–Stokes system with a boundary noise

The restrictions of trajectories of (22), (23) to $\mathbb{Z}_+$ form a random dynamical system in the space

$$H = \{u \in L^2(D, \mathbb{R}^2) : \operatorname{div} u = 0 \text{ in } D, \langle u, \boldsymbol{n} \rangle = 0 \text{ on } \partial D\}.$$

Structure of the noise. $\exists$ *ONB* $\{e_j\}$ *in the space E such that*

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(t, x) \tag{26}$$

where $\{b_j\} \subset \mathbb{R}$ *is a sequence going to zero sufficiently fast and* $\{\xi_{jk}\}$ *are independent random variables such that*

$$\mathcal{D}(\xi_{jk}) = \rho_j(r)\,dr, \quad \rho_j \in C^1(\mathbb{R}), \quad \operatorname{supp} \rho_j \subset [-1, 1].$$

The support $\mathcal{K}$ of the law of η_k is a compact subset in E.

Mixing for the Navier–Stokes system

Theorem

Assume that the following condition holds for some $\hat{u} \in H$:

Approximate controllability to $\hat{u}$: For any $R, \delta > 0$ $\exists\, m \geq 1$ such that, given $u_0 \in B_H(R)$ one can find $\eta_1, \dots \eta_m \in \mathcal{K}$ for which the corresponding solution of (22), (23) satisfies the inequality

$$\|u(k) - \hat{u}\| \leq \delta.$$

In this case, there is $N \geq 1$ depending on $\nu > 0$ such that if

$$b_j \neq 0 \quad \text{for } 1 \leq j \leq N, \tag{27}$$

then $\exists\, C, \gamma > 0$ and a unique measure $\mu \in \mathcal{P}(H)$ such that

$$\|\mathcal{D}_\lambda(u(m)) - \mu\|_L^* \leq C\, e^{-\gamma k}\left(1 + \int_H \|u\|\lambda(du)\right) \quad \forall \lambda \in \mathcal{P}(H).$$

Open problems

Approximate controllability by a bounded localised force

Our result on mixing of the flow for the Navier–Stokes system requires global approximate controllability by a control taking values in the support of η. This condition is trivially satisfied if $h \equiv 0$, and the support of η contains zero. Due to Coron–Fursikov–Imanuvilov (1996–1999), the problem is globally approximately controllable by an *unbounded* C^∞ force. The following question remains completely open:

Nontrivial uncontrolled dynamics

Given a smooth function h, find a compact (or even bounded) subset $\mathcal{K} \subset E$ and a point $\hat{u} \in H$ such that the Navier–Stokes system is globally approximately controllable to $\hat{u}$ with a $\mathcal{K}$-valued control.

Open problems

Squeezing by a bounded Fourier-localised force

Let us consider the Navier–Stokes system with the RHS

$$f(t,x) = h(t,x) + \sum_{j=1}^{N} \eta^j(t) e_j(x), \tag{28}$$

where e_j is an ONB in H. Due to Agrachev–Sarychev (2005), the problem is globally approximately controllable for any $\nu > 0$. The following question is very important when dealing with the problem of mixing for a Fourier-localised noise:

Local stabilisation

Given an arbitrary right-hand side $h(t,x)$ and initial points $v, v' \in H$, construct functions $\eta^1, \dots, \eta^N \in L^2(0,1)$ such that

$$\|u(1) - u'(1)\| \le q\|v - v'\|, \quad \|\eta^j\|_{L^2} \le C\|v - v'\|^\alpha, \tag{29}$$

where $q < 1$, $\alpha \le 1$, and C are some positive numbers.

References

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  Hairer–Mattingly (2006–2011): $h \equiv 0$, $D = \mathbb{T}^2$;
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  Földes–Glatt-Holtz–Richards–Thomann (2015): Boussinesq system with $h \equiv 0$, $D = \mathbb{T}^2$ and noise acting only on the equation for temperature.
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